Stochastic variance reduction for variational inequalities methods

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One World Optimization Seminar

LINKÖPING UNIVERSITY

Intro

- variance reduction in minimization
- variational inequalities (VIs)
- variance reduction in VIs

Joint work with: Ahmet Alacaoglu,
University of Wisconsin-Madison

Reference: arxiv:2102.08352
Support: WASP Program

Empirical risk minimization

$$\min_{x \in \mathbb{R}^d} \left\{ f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x) \right\}$$

Assumption: f is convex and smooth. Both d and N can be large

GD: $x_{k+1} = x_k - \tau \nabla f(x_k)$

SGD: $x_{k+1} = x_k - \tau_k \nabla f_{\xi_k}(x_k)$

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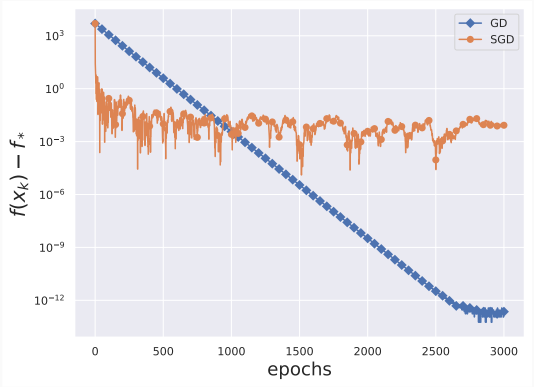
GD: $x_{k+1} = x_k - \tau \nabla f(x_k)$

- fast, expensive iteration

SGD: $x_{k+1} = x_k - \tau_k \nabla f_{\xi_k}(x_k)$

- slow, cheap iteration



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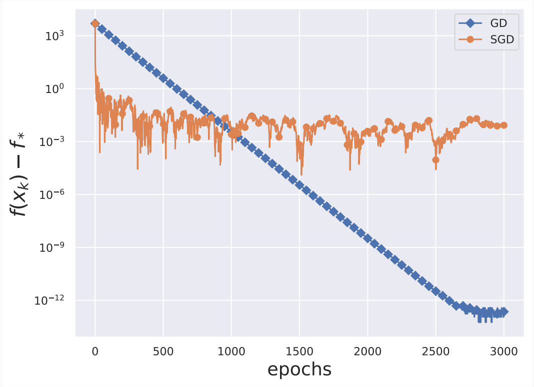

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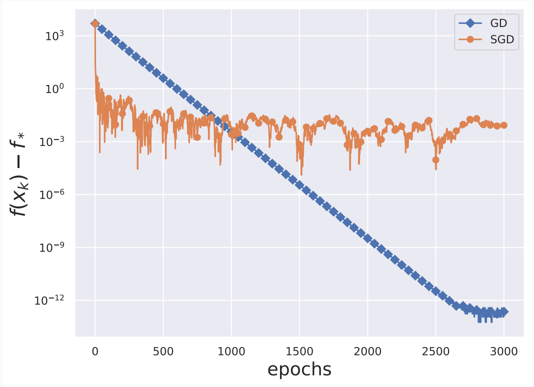
We want the best of both worlds



Naïve idea

SGD: $x_{k+1} = x_k - \tau_k \nabla f_{\xi_k}(x_k)$ Convergence rate: $\mathcal{O}(\frac{1}{\varepsilon^2})$

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Why is it slow?

Because $\nabla f_{\xi_k}(x_k) \neq \nabla f(x_k) \implies \mathrm{Var}[\nabla f_{\xi_k}(x_k)] = \mathbb{E}\left[\|\nabla f_{\xi_k}(x_k) - \nabla f(x_k)\|^2\right] \neq 0$

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Goal: decrease variance.

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Minibatch-SGD:

$$\text{Sample } \mathcal{S}_k \subset \{1, 2, \dots, N\}$$

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Good for convergence, not good for complexity.

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Basic update: $x_{k+1} = x_k - \tau_k g_k$

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$$\nabla f(x_k) - g_k = (\nabla f(x_k) - \nabla f(x_{k-2})) - (\nabla f_{\xi_k}(x_k) - \nabla f_{\xi_k}(x_{k-2})) \to 0$$

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Infrequently compute a full gradient $\boxed{\nabla f(y)}$

SVRG

Superscript x^s for expensive update with $\nabla f(x^s)$

Subscript x_k for cheap update with $\nabla f_{\xi_k}(x_k)$

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[Johnson et al., *NIPS*, 2013]

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Loopless SVRG

Adding another source of randomness removes one loop

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Assumption: f_i is convex and L-smooth; cost of ∇f_i is $\mathcal{O}(1)$
f is L_f-smooth

Complexity table

$$\text{Convergence rate} = \quad \#\text{ iterations to reach } \varepsilon\text{-accuracy: } f(x_k) - f_* \leqslant \varepsilon.$$

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GD: $x_{k+1} = x_k - \tau \nabla f(x_k)$
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Iteration cost	$N \times \mathcal{O}(1)$	$\mathcal{O}(1)$
Convergence rate	$\mathcal{O}\left(\frac{L_f}{\varepsilon}\right)$	$\mathcal{O}\left(\frac{L}{\varepsilon^2}\right)$
Total complexity	$\mathcal{O}\left(\frac{NL_f}{\varepsilon}\right)$	$\mathcal{O}\left(\frac{L}{\varepsilon^2}\right)$

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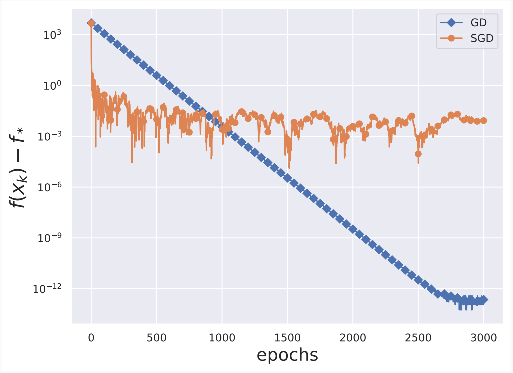
Take-away 1

- VR takes the best from GD and SGD



- References:
  - **SAG**: [Schmidt, et al., *NIPS*, 2013]
  - **SVRG:** [Johnson, et al., *NIPS*, 2013]
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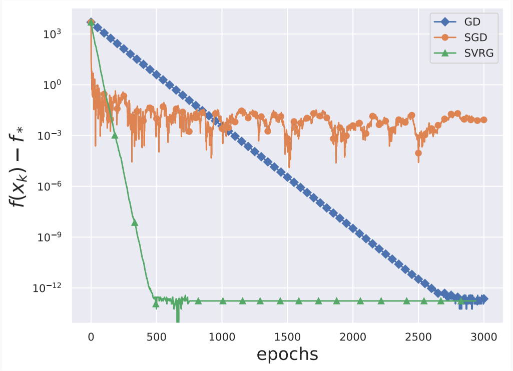
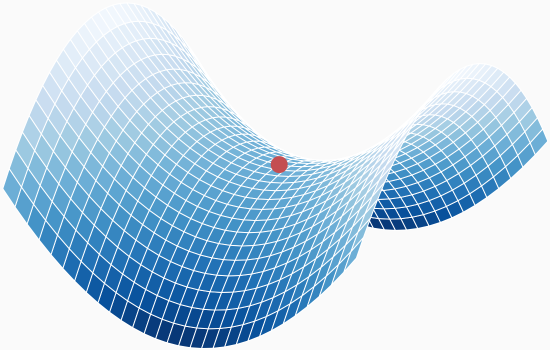


Saddle points



Saddle point problems

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Assumption: F is monotone: $\langle F(z) - F(z'), z - z' \rangle \geqslant 0 \quad \forall z, z' \in \mathcal{Z}$

Motivation

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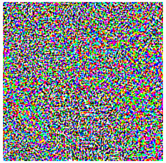
$+\ .007\ \times$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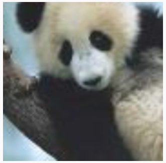
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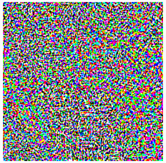
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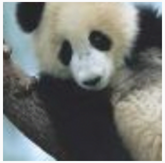
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$$\underbrace{\min_x \sum_i f(x, a_i, b_i)}_{\text{ERM}} \longrightarrow \underbrace{\min_x \max_{\delta \in \mathcal{S}} \sum_i f(x, a_i + \delta, b_i)}_{\text{Robust ERM}}$$

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6. Generative adversarial networks, GANs.
   Two player game between two neural networks.

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Theorem

Let $\tau \in \left(0, \frac{1}{L_F}\right)$. *Then* $z_k \to z_* \in \mathrm{Sol}$ *and* $\mathrm{Gap}(z^K) = \mathcal{O}\left(\frac{L_F}{K}\right)$.

(z^K) is the average of $(z_{k+1/2})$

Finite-sum structure

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) = f_1(x, y) + \cdots + f_N(x, y)$$

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State of the art:

[Balamurugan, Bach, *NIPS*, 2016]: Strongly-convex-strongly-concave case is covered.

[Iusem et al., *SIOPT*, 2017]: With increasing minibatch size.

[Carmon, et al., *NeurIPS*, 2019]: Matrix games, generic case requires unnatural assumptions.

Proposed algorithm

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```
1: **Input:** Probability $p \in (0,1]$, step size $\tau$, $\alpha \in (0,1)$, and $z_0 = w_0$
2: **for** $k = 0, 1, \dots$ **do**
3:     $\bar{z}_k = \alpha z_k + (1-\alpha) w_k$
4:     Sample $\xi_k \in \{1, \dots, N\}$
5:     $z_{k+1/2} = P_{\mathcal{Z}}(\bar{z}_k - \tau F(w_k))$
6:     $z_{k+1} = P_{\mathcal{Z}}(\bar{z}_k - \tau[F(w_k) + F_{\xi_k}(z_{k+1/2}) - F_{\xi_k}(w_k)])$
7:     $w_{k+1} = \begin{cases} z_{k+1}, & \text{with probability } p \\ w_k, & \text{with probability } 1-p \end{cases}$
8: **end for**
```

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When there is no randomness, i.e., $p = 1$ and $F_\xi = F \implies w_{k+1} = z_{k+1} \implies \bar{z}_k = z_k$

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Theorem

Let $p \in (0,1]$, $\alpha = 1 - p$, *and* $\tau \in \left(0, \frac{\sqrt{p}}{L}\right)$. *Then,* $z_k \to z_* \in \mathrm{Sol}$ *a.s. and*

$$\mathbb{E}\left[\mathrm{Gap}(z^K)\right] = \mathcal{O}\left(\frac{L}{\sqrt{p}K}\right).$$

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Deterministic setting

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The same improvement as in the minimization case!

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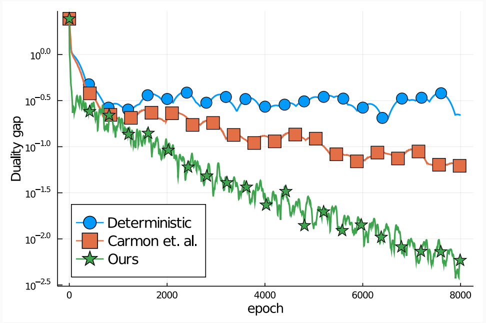
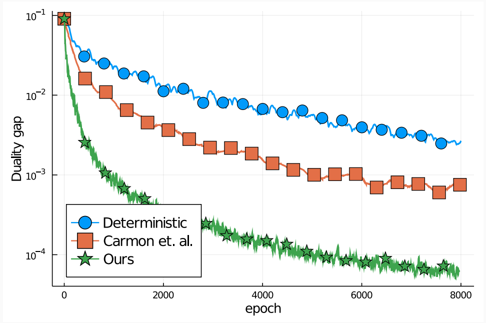
[Grigoriadis & Khachiyan, 1995], [Carmon et al. 2019]

Illustration

$$\min_{x\in\Delta^n}\max_{y\in\Delta^n}\langle Ax, y\rangle$$



'policeman and robber' problem, $n = 500$



random iid matrix from $[0,1]^{n\times n}$, $n = 500$

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Y. Han et al. *"Lower Complexity Bounds of Finite-Sum Optimization Problems: The Results and Construction"* arxiv:2103.08280