The Douglas–Rachford algorithm for inconsistent optimization problems: the complete story

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The One World Optimization Seminar Series 2021
Monday October 4, 2021 via Zoom
15:30–16:15 CEST (9:30–10:15 EDT)

To the organizers ...

Thank you for running this excellent seminar series and for the invitation.

- Danke Radu!
- תודה Shoham!
- Bedankt Mathias!

Our problem ...

Consider the convex optimization problem:

- Find $x \in \mathbb{R}^n$ such that x minimizes the sum of
$$f + g.$$
- The structure $f + g$ suggests splitting methods, e.g., Douglas–Rachford method, etc...
- Our problem: is what if there is no such x, i.e., what if the problem has no solution?
- Consider the problem: Find $x \in \mathbb{R}^n$ such that x minimizes
$$\tfrac{1}{2}\langle Mx \,|\, x\rangle + \langle b \,|\, x\rangle,$$
M is an $n \times n$ positive semidefinite matrix and $b \in \mathbb{R}^n$.
- Fermat's theorem yields the equivalent problem: Find $x \in \mathbb{R}^n$ such that
$$Mx + b = 0.$$
If $b \notin \operatorname{ran} M$ then we have the problem.

The setting

Throughout this talk

X is a real Hilbert space

with inner product $\langle \cdot \mid \cdot \rangle$, and induced norm $\|\cdot\|$, e.g., $\mathbb{R}^n$, $\mathbb{S}^n$ or ℓ^2.

- Recall that an operator $A\colon X \rightrightarrows X$ is monotone if

$$\{(x,u),(y,v)\} \subseteq \operatorname{gr} A \Rightarrow \langle x-y \mid u-v \rangle \geq 0.$$

- Recall also that a monotone operator A is maximally monotone if A cannot be properly extended without destroying monotonicity.
- Examples: Matrices with positive semidefinite parts, subdifferential operators ∂f of convex functions and skew symmetric operators, e.g.,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The problem: a more general formulation

Throughout the talk we assume that

A and B are maximally monotone operators on X.

The problem:
Find $x \in X$ such that

$$\text{(P)} \qquad x \in \operatorname{zer}(A + B) = \{x \in X \mid 0 \in Ax + Bx\}.$$

The Douglas–Rachford algorithm: One successful technique to find a zero of $A + B$ is via iterating the Douglas–Rachford operator $T_{A,B}$ defined for the ordered pair (A, B) by

$$T_{A,B} = \tfrac{1}{2}(\mathrm{Id} + \mathrm{R}_B \mathrm{R}_A).$$

• $\mathrm{Id}\colon X \to X\colon x \mapsto x$. • $\mathrm{R}_A := 2\mathrm{J}_A - \mathrm{Id} = 2(\mathrm{Id} + A)^{-1} - \mathrm{Id}$.

Motivation

The problem: the differential form.

(P) Find $x \in \mathbb{R}^n$ such that x minimizes $f + g$.

- Suppose that f and g are smooth. Then (P) is equivalent to

$$\text{find } x \in X \text{ such that } 0 = \nabla(f+g)(x) = \nabla f(x) + \nabla g(x).$$

- If we drop the assumption of smoothness, (P) reduces to

$$\text{find } x \in X \text{ such that } 0 \in \partial(f+g)(x) = \partial f(x) + \partial g(x),$$

where $\partial f(x) = \left\{ u \in \mathbb{R}^n \;\middle|\; (\forall y)\ \langle u, y - x \,|\, + \rangle f(x) \leq f(y) \right\}$.

Example of Constraint Qualifications (CQs): • $\operatorname{dom} f \cap \operatorname{int} \operatorname{dom} g \neq \varnothing$.

Examples

The problem:

(P) Find $x \in \mathbb{R}^n$ such that x minimizes $f + g$.

Let U be a nonempty closed convex subset of X. Recall that the indicator function of U, denoted by ι_U, is defined by

$$\iota_U(x) = \begin{cases} 0, & x \in U; \\ +\infty, & \text{otherwise.} \end{cases}$$

- Constrained convex optimization problem:

  $$\left.\begin{array}{l} \text{minimize } f(x) \\ \text{subject to } x \in U \end{array}\right\} \longrightarrow \text{find } x \in \mathbb{R}^n \text{ such that } x \text{ minimizes } f + \iota_U.$$

- Convex feasibility problem:

  find x such $x \in U \cap V \longrightarrow$ find $x \in \mathbb{R}^n$ such that x minimizes $\iota_U + \iota_V$.

Classical convergence results

Let $x_0 \in X$. Recall that when

$$\operatorname{zer}(A+B) = \{x \in X \mid 0 \in Ax + Bx\} \neq \varnothing$$

we have:

- Lions–Mercier (1979)

$$x_n = T^n x_0 \xrightarrow{\text{weakly}} \text{some point } \overline{x} = T\overline{x} \in \operatorname{Fix} T \neq \operatorname{zer}(A+B).$$

- Combettes (2004) $\mathrm{J}_A(\operatorname{Fix} T) = \operatorname{zer}(A+B)$. Consequently,

$$\operatorname{Fix} T \neq \varnothing \Leftrightarrow \operatorname{zer}(A+B) \neq \varnothing.$$

- Svaiter (2009)

$$\mathrm{J}_A T^n x_0 \xrightarrow{\text{weakly}} \mathrm{J}_A \overline{x} \in \operatorname{zer}(A+B).$$

• $\mathrm{J}_A := (\operatorname{Id} + A)^{-1}$. • $\mathrm{R}_A := 2\mathrm{J}_A - \operatorname{Id}$. • $T := \operatorname{Id} - \mathrm{J}_A + \mathrm{J}_B \mathrm{R}_A$.

Classical convergence results: function version

Let $x_0 \in X$. Recall that when

$$\operatorname{zer}(\partial f + \partial g) = \{x \in X \mid \partial f(x) + \partial g(x)\} \neq \varnothing$$

we have:

- Lions–Mercier (1979)

$$x_n = T^n x_0 \xrightarrow{\text{weakly}} \text{some point } \overline{x} = T\overline{x} \in \operatorname{Fix} T \neq \operatorname{zer}(\partial f + \partial g).$$

- Combettes (2004) $\operatorname{Prox}_f(\operatorname{Fix} T) = \operatorname{zer}(\partial f + \partial g)$. Consequently,

$$\operatorname{Fix} T \neq \varnothing \Leftrightarrow \operatorname{zer}(\partial f + \partial g) \neq \varnothing.$$

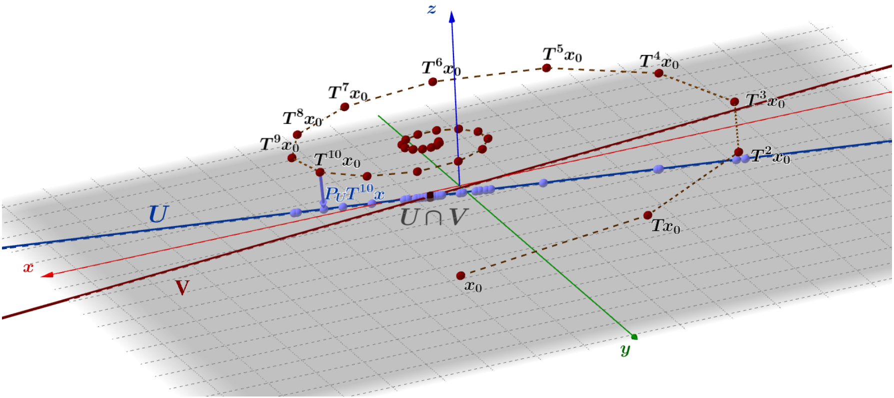
- Lions–Mercier–Svaiter

$$\operatorname{Prox}_f T^n x \xrightarrow{\text{weakly}} \text{some point in } \operatorname{argmin}(f+g).$$

$\operatorname{Prox}_f(x) = \operatorname{argmin}_{y \in X} \left(f(y) + \frac{1}{2}\|x - y\|^2 \right).$

DR for two lines in $\mathbb{R}^3$

$$f = \iota_U,\ g = \iota_V \text{ and } T = \frac{1}{2}\Big(\mathrm{Id} + (2P_V - \mathrm{Id}) \circ (2P_U - \mathrm{Id})\Big).$$



U = the blue line,
V = the red line,
$(T^n x_0)_{n\in\mathbb{N}}$ = the red sequence,
$(P_U T^n x_0)_{n\in\mathbb{N}}$ = the blue sequence.

Convergence results: what if?

Let $x_0 \in X$. Recall that when

$$\operatorname{zer}(\partial f + \partial g) = (\partial f + \partial g)^{-1}(0) \neq \varnothing$$

we have:

- Lions–Mercier (1979)

$$x_n = T^n x_0 \xrightarrow{\text{weakly}} \text{some point } \overline{x} = T\overline{x} \in \operatorname{Fix} T \neq \operatorname{zer}(\partial f + \partial g).$$

- Combettes (2004) $\operatorname{Prox}_f(\operatorname{Fix} T) = \operatorname{zer}(\partial f + \partial g)$. Consequently,

$$\operatorname{Fix} T \neq \varnothing \Leftrightarrow \operatorname{zer}(\partial f + \partial g) \neq \varnothing.$$

- Lions–Mercier–Svaiter

$$\operatorname{Prox}_f T^n x \xrightarrow{\text{weakly}} \text{some point in } \operatorname{argmin}(f+g).$$

- Question: What happens when $\operatorname{zer}(\partial f + \partial g) = \varnothing$?

$\operatorname{Prox}_f(x) = \operatorname{argmin}_{y \in X} \left(f(y) + \frac{1}{2}\|x - y\|^2 \right).$

The case of infeasible affine subspaces: Example

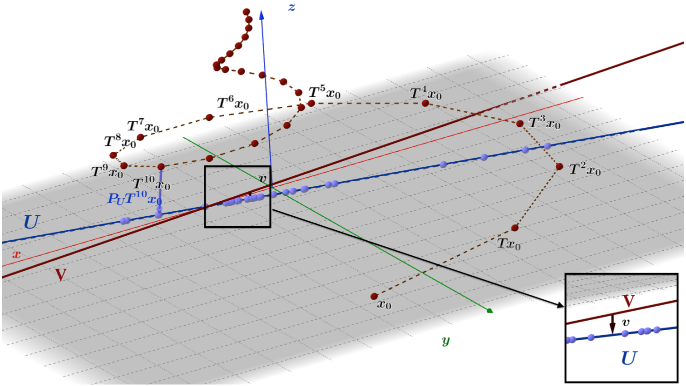


Figure: A GeoGebra snapshot. Two nonintersecting affine subspaces U (blue line) and V (red line) in $\mathbb{R}^3$. Shown are also the first few iterates of $(T^n x_0)_{n\in\mathbb{N}}$ (red points) and $(P_U T^n x_0)_{n\in\mathbb{N}}$ (blue points). In this case $\|T^n x_0\| \to +\infty$ but $(P_U T^n x_0)_{n\in\mathbb{N}}$ remains bounded!

The generalized framework of the normal problem: the right tools

- The minimal displacement vector

$$v := \mathrm{P}_{\overline{\mathrm{ran}}(\mathrm{Id}-T)}(0).$$

- The normal problem: Find $x \in X$ such that

$$x \in \mathrm{zer}\big(-v + A + B(\cdot - v)\big).$$

- The generalized solution set or the *normal solutions*

$$Z = \big\{x \in X \;\big|\; 0 \in -v + Ax + B(x - v)\big\}.$$

Roots in linear algebra: least squares

- Suppose that $X = \mathbb{R}^n$, let $A \in \mathbb{R}^{n\times n}$ be such that $A + A^{\mathsf{T}}$ is positive semidefinite (A is maximally monotone!).
- Find $x \in \mathbb{R}^n$ such that $Ax = b$. Set $B \equiv -b$. The problem reduces to: Find $x \in \mathbb{R}^n$ such that
$$x \in \operatorname{zer}(A + B).$$
- If $b \notin \operatorname{ran} A$ then we $\operatorname{zer}(A + B) = \varnothing$.
- The minimal displacement vector is
$$v = -\mathrm{P}_{(\operatorname{ran} A)^\perp}(b).$$
- The normal solutions are the least squares solutions!

Earlier works

Let $x_0 \in X$. When $\operatorname{zer}(A+B) = \varnothing$, equivalently, $\operatorname{Fix} T = \varnothing$, we always have

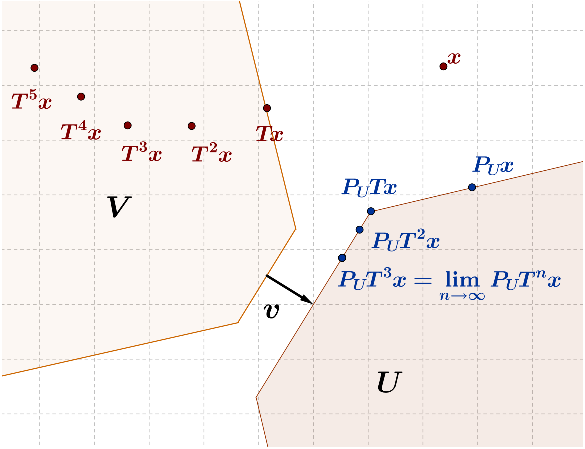
$$\|T^n x_0\| \to \infty.$$

Suppose that

$$v \in \operatorname{ran}(\mathrm{Id} - T).$$

- Bauschke–Combettes–Luke (2003) proved that when $(f, g) = (\iota_U, \iota_V)$, U, V nonempty closed convex subsets of X, then the shadow sequence $(\mathrm{P}_U T^n x)_{n\in\mathbb{N}}$ is bounded and its weak cluster points are minimizers of the function $\iota_U + \iota_V(\cdot - v)$ (i.e., normal solutions!).
- Bauschke–M (2015) proved the strong convergence of the shadow sequence with a linear rate and identified the limit when U, V are closed affine subspaces.
- Bauschke–Dao–M (2015) & Bauschke–M (2016) proved the weak convergence of the shadow sequence to a normal solution when U, V nonempty closed convex subsets of X.
- Bauschke–M (2019) proved the weak convergence of the shadow sequence to a normal solution when f is convex lower semicontinuous and proper and $g = \iota_U$ where U is a closed affine subspace X under the assumption that $0 \in \operatorname{dom} f^* + U^\perp$.

Convex feasibility example



A GeoGebra snapshot. U and V are two nonintersecting sets in $\mathbb{R}^2$. Also, the first few iterates of the governing sequence $(T^n x)_{n\in\mathbb{N}}$ (red points) and the shadow sequence $(P_U T^n x)_{n\in\mathbb{N}}$ (blue points) are shown.

Related works

Of central importance of these results was the following fact:

- Bauschke–Hare–M (2014): Suppose X is finite-dimensional and A and B are nice, e.g., subdifferentials of convex functions f and g respectively. Then
$$\overline{\operatorname{ran}}(\mathrm{Id} - T) = \overline{\operatorname{dom} f - \operatorname{dom} g} \cap \overline{\operatorname{dom} f^* + \operatorname{dom} g^*}.$$
- Ryu–Lin–Yin (2017 and 2018 respectively) proposed a method based on the Douglas–Rachford algorithm that identifies, in certain situations, infeasible, unbounded, and pathological conic (and feasible and infeasible convex, respectively) optimization problems.
- Banjac–Goulart–Stellato–Boyd (2018) showed that for certain classes of convex optimization problems, ADMM can detect primal and dual infeasibility of the problem and they propose a termination criterion.
- Banjac–Lygeros and Banjac (2020) extended some of the geometric properties of the minimal displacement vector established in our 2019 work.

More generally ...

Let $x_0 \in X$.

- Can we learn more when A and B are nice maximally monotone operators?
- As a first step: Can we characterize when the shadows are bounded?
- Suppose the shadows are bounded. Can we locate the weak cluster points? What about full convergence??

Our assumptions: A1

We assume that

$$\overline{\operatorname{ran}}(\operatorname{Id} - T) = \overline{\operatorname{dom} A - \operatorname{dom} B} \cap \overline{\operatorname{ran} A + \operatorname{ran} B}.$$

True, e.g., when X is finite-dimensional and $(A, B) = (\partial f, \partial g)$.

A1 holds in the optimization settings when X is finite-dimensional. ✓

The beautiful geomerty and the vectors v_D and v_R!

Recall that $\overline{\operatorname{ran}}(\operatorname{Id} - T) = \overline{\operatorname{dom} A - \operatorname{dom} B} \cap \overline{\operatorname{ran} A + \operatorname{ran} B}$. We now introduce the vectors

$$v_D := P_{\overline{\operatorname{dom} A - \operatorname{dom} B}}(0) \quad \text{and} \quad v_R := P_{\overline{\operatorname{ran} A + \operatorname{ran} B}}(0)$$

We can conclude

(i) $v_D \in (-\operatorname{rec} \overline{\operatorname{dom}}\, A)^{\ominus} \cap (\operatorname{rec} \overline{\operatorname{dom}}\, B)^{\ominus}$.

(ii) $v_R \in (-\operatorname{rec} \overline{\operatorname{ran}}\, A)^{\ominus} \cap (-\operatorname{rec} \overline{\operatorname{ran}}\, B)^{\ominus}$.

• $\operatorname{rec} C = \{x \in X \mid x + C \subseteq C\}$. • $C^{\ominus} = \{u \in X \mid \sup\langle C \mid u\rangle \leq 0\}$.

Fact

Let U and V be nonempty closed convex subsets of X. Then

$$P_{\overline{U-V}}(0) \in \overline{(P_U - \operatorname{Id})(V)} \cap \overline{(\operatorname{Id} - P_V)(U)} \subseteq (-\operatorname{rec} U)^{\ominus} \cap (\operatorname{rec} V)^{\ominus}.$$

The beautiful geomerty

The following lemma is of crucial importance in our work.

Lemma

The following hold for A and B:

(i) $(\operatorname{rec}\overline{\operatorname{dom}}A)^{\ominus} \subseteq \operatorname{rec}(\overline{\operatorname{ran}}A)$ *and* $(\operatorname{rec}\overline{\operatorname{dom}}B)^{\ominus} \subseteq \operatorname{rec}(\overline{\operatorname{ran}}B)$.

(ii) $(\operatorname{rec}\overline{\operatorname{ran}}A)^{\ominus} \subseteq \operatorname{rec}(\overline{\operatorname{dom}}A)$ *and* $(\operatorname{rec}\overline{\operatorname{ran}}B)^{\ominus} \subseteq \operatorname{rec}(\overline{\operatorname{dom}}B)$.

Proof.

Using the celebrated Brezis–Haraux theorem

$$\overline{\operatorname{ran}}\,A + \overline{\operatorname{ran}}\,\mathrm{N}_{\overline{\operatorname{dom}}A} \subseteq \overline{\operatorname{ran}\,A + \operatorname{ran}\,\mathrm{N}_{\overline{\operatorname{dom}}A}} = \overline{\operatorname{ran}}(A + \mathrm{N}_{\overline{\operatorname{dom}}A}) = \overline{\operatorname{ran}}\,A$$

and we conclude that

$$\overline{\operatorname{ran}}\,\mathrm{N}_{\overline{\operatorname{dom}}A} \subseteq \operatorname{rec}\overline{\operatorname{ran}}A.$$

On the other hand, using a result by Zarantonello we have

$$\overline{\operatorname{ran}}\,\mathrm{N}_{\overline{\operatorname{dom}}A} = \overline{\operatorname{ran}}\,(\mathrm{Id} - \mathrm{P}_{\overline{\operatorname{dom}}A}) = (\operatorname{rec}\overline{\operatorname{dom}}A)^{\ominus}.$$

□

• $\operatorname{rec} C = \{x \in X \mid x + C \subseteq C\}$. • $C^{\ominus} = \{u \in X \mid \sup\langle C \mid u\rangle \leq 0\}$.

The beautiful geomerty and locating v_D and v_R!

Proposition

The following hold:

(i) $\langle v_D \mid v_R \rangle = 0$.

(ii) $v = v_D + v_R$.

• $v_D := P_{\overline{\operatorname{dom} A - \operatorname{dom} B}}(0)$. • $v_R := P_{\overline{\operatorname{ran} A + \operatorname{ran} B}}(0)$.

Dynamic consequences

Known: Let $x \in X$. Then

$$\mathrm{J}_A T^n x - \mathrm{J}_B \mathrm{R}_A T^n x = \mathrm{J}_{A^{-1}} T^n x + \mathrm{J}_{B^{-1}} \mathrm{R}_A T^n x = T^n x - T^{n+1} x \to v.$$

Proposition

Let $x \in X$. Then the following hold:

(i) $\mathrm{J}_A T^n x - \mathrm{J}_A T^{n+1} x \to v_R$.

(ii) $\mathrm{J}_{A^{-1}} T^n x - \mathrm{J}_{A^{-1}} T^{n+1} x \to v_D$.

Proposition (**shadow convergence: necessary condition)**

Let $x \in X$. Then the following hold:

(i) $(\mathrm{J}_A T^n x)_{n \in \mathbb{N}}$ *is asymptotically regular* $\Leftrightarrow v_R = 0$.

(ii) $(\mathrm{J}_{A^{-1}} T^n x)_{n \in \mathbb{N}}$ *is asymptotically regular* $\Leftrightarrow v_D = 0$.

• $\mathrm{Id} - T = \mathrm{J}_A - \mathrm{J}_B \mathrm{R}_A = \mathrm{J}_{A^{-1}} + \mathrm{J}_{B^{-1}} \mathrm{R}_A$. • $v_D := \mathrm{P}_{\overline{\mathrm{dom}\, A - \mathrm{dom}\, B}}(0)$. • $v_R := \mathrm{P}_{\overline{\mathrm{ran}\, A + \mathrm{ran}\, B}}(0)$. • $v = v_D + v_R$.

Our assumptions: A2

We assume that

$$v \in \mathrm{ran}(\mathrm{Id} - T).$$

Equivalently (proof omitted),

$$Z = \{x \in X \mid 0 \in -v + Ax + B(x - v)\} \neq \varnothing.$$

- $v = P_{\overline{\mathrm{ran}}(\mathrm{Id} - T)}(0)$.

And finally we see Fejér monotonicity!

Working in $X \times X$ we state the following key result:

Theorem

Suppose that $v \in \operatorname{ran}(\operatorname{Id} - T)$, let $x \in X$, and let $n \in \mathbb{N}$. Then the following hold:

(i) *Suppose that A and B are paramonotone (true when $(A, B) = (\partial f, \partial g)$). Then the sequence*

$$\big((0, -v) + (J_A T^n x + n v_R, J_{A^{-1}} T^n x + n v_D)\big)_{n\in\mathbb{N}}$$

is Fejér monotone with respect to $Z \times K$.

(ii) *The sequence $(J_A T^n x + n v_R, J_{A^{-1}} T^n x + n v_D)_{n\in\mathbb{N}}$ is bounded.*

(iii) *The sequence $(J_A T^n x)_{n\in\mathbb{N}}$ is bounded $\Leftrightarrow v_R = 0$.*

(iv) *The sequence $(J_{A^{-1}} T^n x)_{n\in\mathbb{N}}$ is bounded $\Leftrightarrow v_D = 0$.*

• $Z := \operatorname{zer}(-v + A + B(\cdot - v))$. • $K := \operatorname{zer}((-v + A)^{-1} + (B(\cdot - v))^{-\circledcirc})$.
• $A^{-\circledcirc} = (-\operatorname{Id}) \circ A^{-1} \circ (-\operatorname{Id})$.

The optimization setting

From now on we assume that

$$f \text{ and } g \text{ are proper lsc convex functions on } X$$

and that $(A, B) = (\partial f, \partial g)$. And finally we assume (A3):

$$v_R = 0 \quad \Leftrightarrow \quad v = \mathrm{P}_{\overline{\mathrm{ran}}(\mathrm{Id} - T)}(0) = \mathrm{P}_{\overline{\mathrm{dom}\, f - \mathrm{dom}\, g}}(0) = v_D.$$

We use the abbreviations

$$\big(P_f, P_{f^*}, P_g, R_f\big) = \big(\operatorname{Prox}_f, \operatorname{Prox}_{f^*}, \operatorname{Prox}_g, 2\operatorname{Prox}_f - \mathrm{Id}\big).$$

Hence

$$T = T_{(\partial f, \partial g)} = \mathrm{Id} - P_f + P_g R_f.$$

• $v_D := \mathrm{P}_{\overline{\mathrm{dom}\, A - \mathrm{dom}\, B}}(0)$. • $v_R := \mathrm{P}_{\overline{\mathrm{ran}\, A + \mathrm{ran}\, B}}(0)$. • $v = v_D + v_R$. • $\mathrm{J}_{\partial f} = P_f$. • $\mathrm{J}_{(\partial f)^{-1}} = \mathrm{J}_{\partial f^*} = P_{f^*}$.

Convergence proof in a nutshell

- Step 1: refining Z.
  Because $v_R = 0$ we learn that $Z = \{x \in X \mid 0 \in \partial f(x) + \partial g(x - v)\}$.
  Because $Z \neq \varnothing$ (recalling (A2) $v \in \operatorname{ran}(\mathrm{Id} - T)$) we prove that
  $Z = \operatorname{argmin}(f + g(\cdot - v))$.
- Step 2: boundedness of the shadows.
  This is a consequence of Fejér monotonicity and the assumption $v_R = 0$.
- Step 3: locating the weak cluster points of the shadows.
  We show that the weak cluster points are minimizers of $f + g(\cdot - v)$.
- Step 4: full weak convergence of the shadows.
  We combine Step 2, Step 3 and properties of Fejér monotone sequences.

Step 1: refining Z

Proposition

Recalling $Z = \{x \in X \mid 0 \in -v + \partial f(x) + \partial g(x - v)\}$*, and* $v_R = 0$*, we have:*

(i) $Z = \{x \in X \mid 0 \in \partial f(x) + \partial g(x - v)\}$.

(ii) $Z \neq \varnothing \Rightarrow Z = \operatorname{argmin}_{x \in X}(f(x) + g(x - v))$.

• $v_D := P_{\overline{\operatorname{dom} A - \operatorname{dom} B}}(0)$. • $v_R := P_{\overline{\operatorname{ran} A + \operatorname{ran} B}}(0)$. • $v = v_D + v_R$.

Step 2: boundedness of the shadows

We proved earlier that: the sequence

$$\big((0, -v) + (\mathrm{J}_A T^n x + n v_R, \mathrm{J}_{A^{-1}} T^n x + n v_D)\big)_{n \in \mathbb{N}}$$

is Fejér monotone with respect to $Z \times K$.
Using that $(A, B, v_R) = (\partial f, \partial g, 0)$ we have $(\mathrm{J}_A, \mathrm{J}_{A^{-1}}) = (P_f, P_{f^*})$ and therefore the sequence

$$\big((0, -v) + (P_f T^n x, P_{f^*} T^n x + n v)\big)_{n \in \mathbb{N}}$$

is Fejér monotone with respect to $Z \times K$.

• $v_D := \mathrm{P}_{\overline{\mathrm{dom}\, A - \mathrm{dom}\, B}}(0)$. • $v_R := \mathrm{P}_{\overline{\mathrm{ran}\, A + \mathrm{ran}\, B}}(0)$. • $v = v_D + v_R$. • $Z := \mathrm{zer}(\partial f + \partial g(\cdot - v))$. • $K := \mathrm{zer}((\partial f)^{-1} + (\partial g(\cdot - v))^{-\circledcirc})$.

Step 3: locating the weak cluster points of the shadows.

Proposition

Set $\mu := \min_{x \in X}(f(x) + g(x - v))$ *and let* $x \in X$. *Then the following hold:*

(i) $(P_f T^n x)_{n\in\mathbb{N}}$ *is bounded and its weak cluster points are minimizers of* $f + g(\cdot - v)$.

(ii) $(P_g R_f T^n x)_{n\in\mathbb{N}}$ *is bounded and its weak cluster points are minimizers of* $f(\cdot + v) + g$.

Now let $\overline{z}$ *be a weak cluster point of* $(P_f T^n x)_{n\in\mathbb{N}}$. *Then:*

(iii) $f(P_f T^n x) \to f(\overline{z})$. *(value convergence ✓)*

(iv) $g(P_g R_f T^n x) \to g(\overline{z} - v)$.

(v) $f(P_f T^n x) + g(P_g R_f T^n x) \to \mu$.

Step 4: full weak convergence of the shadows.

Proposition

Let $x \in X$. Then the following hold:

(i) *The sequence $(P_f T^n x)_{n\in\mathbb{N}}$ converges weakly to a minimizer of $f + g(\cdot - v)$.*

(ii) *The sequence $(P_g R_f T^n x)_{n\in\mathbb{N}}$ converges weakly to a minimizer of $f(\cdot + v) + g$.*

Proof.

(i) We showed that the sequence $(P_f T^n x, -v + P_{f^*} T^n x + nv)_{n\in\mathbb{N}}$ is Fejér monotone with respect to $Z \times K$. Now let z_1 and z_2 be two weak cluster points of $(P_f T^n x)_{n\in\mathbb{N}}$. On the one hand,

$$\{z_1, z_2\} \subseteq \operatorname*{argmin}_{x\in X}(f + g(\cdot - v)) = Z; \quad \text{hence,} \quad z_1 - z_2 \in Z - Z.$$

On the other hand, $z_1 - z_2 \in (Z - Z)^\perp$ (proof omitted). Altogether we conclude that $z_1 - z_2 \in (Z - Z) \cap (Z - Z)^\perp = \{0\}$. Hence, $z_1 = z_2$.✓

(ii) A direct consequence of (i) and earlier result. □

How critical are our assumptions?

- (A1)
$$\overline{\operatorname{ran}}(\operatorname{Id} - T) = \overline{\operatorname{dom} A - \operatorname{dom} B} \cap \overline{\operatorname{ran} A + \operatorname{ran} B}.$$
  True, e.g., when X is finite-dimensional and $(A, B) = (\partial f, \partial g)$.
  A1 holds in the optimization settings when X is finite-dimensional. .✓
- (A3) $v_R = 0$. We have proved that it is a necessary and sufficient condition for convergence.✓
- (A2) $v \in \operatorname{ran}(\operatorname{Id} - T)$.

A2 fails and the shadows converge in one step!

- Suppose that $X = \mathbb{R}$.
- Set $(f, g) = (\iota_{\{0\}}, -\sqrt{\cdot})$.
- Clearly, $\operatorname{dom} \partial f = \operatorname{dom} N_{\{0\}} = \{0\}$, $\operatorname{dom} \partial g =]0, +\infty[$. Moreover, $\operatorname{ran} \partial f = \mathbb{R} = \operatorname{ran} \partial f + \operatorname{ran} \partial g$.
- Hence, $Z = \varnothing$.
- $\overline{\operatorname{ran}}(\mathrm{Id} - T) = \overline{\operatorname{dom} \partial f - \operatorname{dom} \partial g} = [0, +\infty[$ and $v = 0 \notin \operatorname{ran}(\mathrm{Id} - T)$.
- $(\forall n \in \mathbb{N})\ P_f T^n x = P_{\{0\}} T^n x = 0$. ✓

A2 fails and the shadows are unbounded.

We revisit an example by Ryu–Liu–Yin (2019).

- Suppose that $X = \mathbb{R}^3$.
- Let $K = \left\{(x_1, x_2, x_3) \;\middle|\; \sqrt{x_1^2 + x_2^2} \leq |x_3|\right\}$
- Set $(f, g) = (\iota_K, \langle e_1 \mid \cdot\rangle + \iota_{\{x_2 = x_3\}})$.
- Let $x \in \mathbb{R} \times \mathbb{R} \times \{0\}$.
- After filling in a lot of details ....
- $v = 0 \notin \operatorname{ran}(\operatorname{Id} - T)$.
- $Z = \varnothing$.
- $\operatorname{argmin}(f + g) = K \cap (\mathbb{R} \cdot (0, 1, 1)) \neq \varnothing$.
- $(\forall n \in \mathbb{N})\ \|P_f T^n x_0\| = \|P_K T^n x\| \to +\infty$.✓

References

H.H. Bauschke and W.M. Moursi (2021). On the Douglas–Rachford algorithm for solving possibly inconsistent optimization problems, https://arxiv.org/pdf/2106.11547.pdf

THANK YOU!!

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