Back to Single-Resolvent Iterations, with Warping

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Outline

- Part 1: Background
- Part 2: The warped resolvent
- Part 3: Warped proximal iterations in Hilbert spaces
- Part 4: Warped proximal iterations with Bregman kernels

Background

PART 1:

Background

Monotone operator splitting in Hilbert spaces

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- Considerable range of applications: optimization,
  - Subdifferential: $M = \partial f$ (Fermat's rule)
  - Kuhn-Tucker operator: $M = \begin{bmatrix} \partial f & L^* \\ -L & \partial g^* \end{bmatrix}$.

    (Rockafellar 1967)
  - etc. (Eckstein 1994, PLC 2018, Bùi/PLC 2020).

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Monotone operator splitting in Hilbert spaces

- Basic problem: Given a maximally monotone operator $M\colon \mathcal{X} \to 2^{\mathcal{X}}$, find $x \in \mathcal{X}$ such that $0 \in Mx$.
- The proximal point algorithm (Bellman 1966, Martinet 1970, Rockafellar 1976):

$$x_{n+1} = J_M x_n, \text{ where } J_M = (\mathrm{Id} + M)^{-1} \text{ is the resolvent of } M.$$

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- Acknowledging the fact that J_M may be hard to implement, *splitting methods* have been developed: the goal is to express M as a combination of operators, and devise an algorithm that uses these operators individually.

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- The following structures have been considered:

$$M = A + B$$

(Mercier 1979, Lions/Mercier 1979, Tseng 2000)

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$$M = \sum_{k=1}^{p} A_k$$

(Spingarn 1983, Gol'stein 1985, Eckstein/Svaiter 2009, PLC 2009)

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$$M = \sum_{k=1}^{p} L_k^* \circ B_k \circ L_k$$

(Briceño-Arias/PLC 2011)

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- The following structures have been considered:

$$M = A + \sum_{k=1}^{p} L_k^* \circ (B_k \,\square\, D_k) \circ L_k + C$$

(PLC/Pesquet 2012, Vũ 2013, Condat 2013, Boţ/Hendrich 2013)

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(Raguet/Fadili/Peyré 2013, Briceño-Arias 2015, Davis/Yin 2017, Raguet 2019)

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- The following structures have been considered:

$$M\colon (x_1, \ldots, x_m) \mapsto \bigtimes_{i=1}^{m} \Bigg(A_i x_i + C_i x_i + Q_i x_i + \sum_{k=1}^{p} L_{ki}^{*}\bigg(\Big(\big(B_k^m + B_k^c + B_k^l\big) \,\square\, \big(D_k^m + D_k^c + D_k^l\big)\Big)\Big(\sum_{j=1}^{m} L_{kj} x_j\Big)\bigg)\Bigg)$$

(Bùi/PLC 2020)

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- ... which models in particular

$$\underset{x_1 \in \mathcal{X}_1, \ldots, x_m \in \mathcal{X}_m}{\text{minimize}} \quad \sum_{i=1}^{m} \big(f_i(x_i) + \varphi_i(x_i)\big) + \sum_{k=1}^{p} \big((g_k + \psi_k) \,\square\, h_k\big)\Bigg(\sum_{j \in I} L_{kj} x_j\Bigg).$$

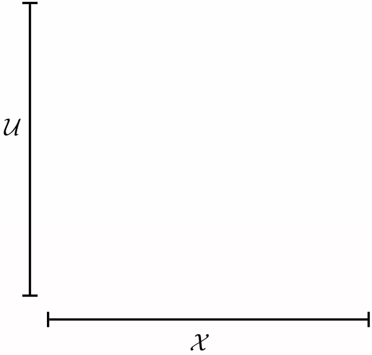
(Bùi/PLC 2020)

Monotone operator splitting

- The field has evolved in many exciting directions and various algorithms are now available for complex structured problems, together with block-coordinate, block-iterative, and asynchronous implementations.
- A common feature of these developments is to move away from single-resolvent iterations such as the proximal point algorithm.
- We introduce an extended notion of a resolvent, called warped resolvent, and show that considering the warped resolvent iterations of a single operator provides a surprisingly broad platform to not only recover existing schemes in a synthetic framework, but also design new ones.

Set-valued operators

- $\mathcal{X}$ and $\mathcal{U}$ nonempty sets, $2^{\mathcal{U}}$ the power set of $\mathcal{U}$.



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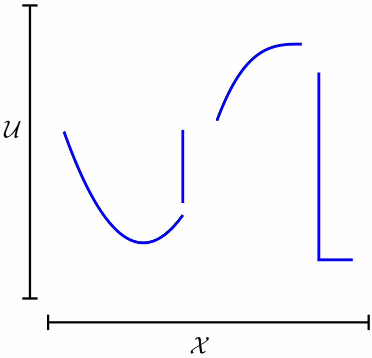
- $M\colon \mathcal{X} \to 2^{\mathcal{U}} : x \mapsto Mx \subset \mathcal{U}$ a set-valued operator.



graph of M: gra $M = \big\{(x,u) \in \mathcal{X} \times \mathcal{U} \mid u \in Mx\big\}$.

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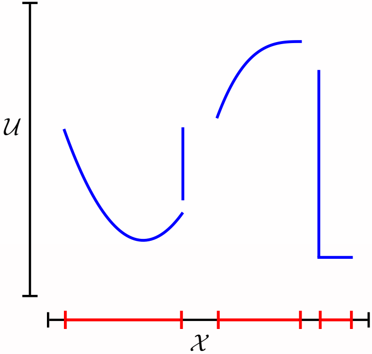

domain of M: dom $M = \{x \in \mathcal{X} \mid Mx \neq \varnothing\}$.

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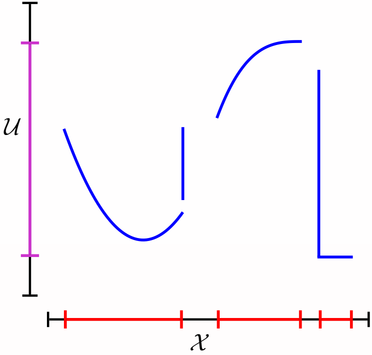

range of M: ran $M = \bigcup_{x\in \operatorname{dom} M} Mx$.

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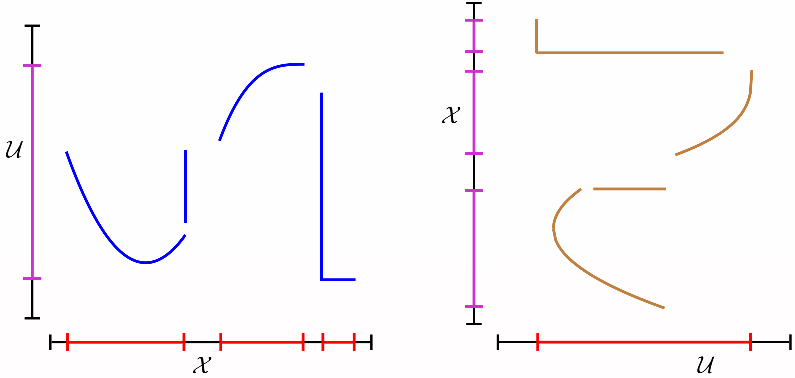

inverse of M: gra $M^{-1} = \{(u,x) \in \mathcal{U} \times \mathcal{X} \mid u \in Mx\}$.

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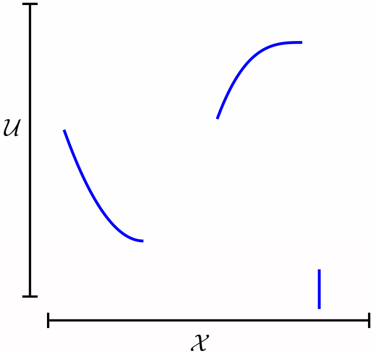

M is injective if : $(\forall x \in \mathcal{X})(\forall y \in \mathcal{X})\ \ Mx \cap My \neq \varnothing \ \ \Rightarrow \ \ x = y.$

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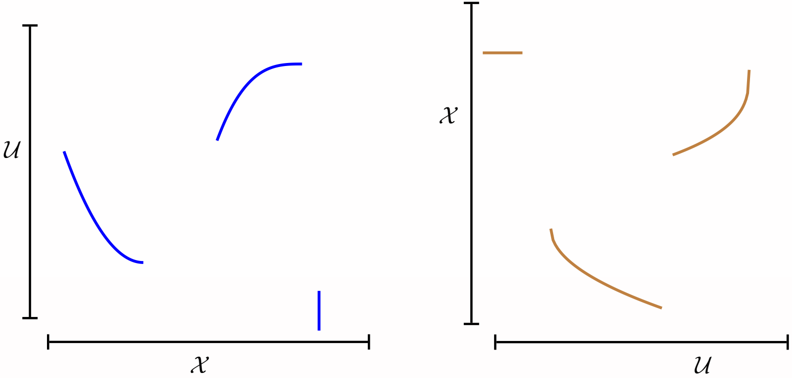

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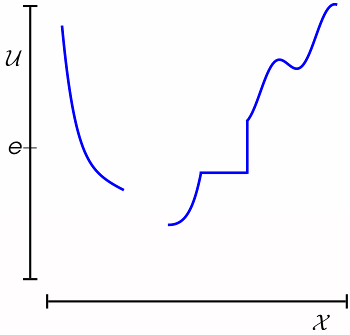
This implies that M^{-1} is at most single-valued.

Problem model: Solving set-valued inclusions

- $\mathcal{X}$ a set, $(\mathcal{U}, \boxplus)$ a group with identity e, $M\colon \mathcal{X} \to 2^{\mathcal{U}}$.



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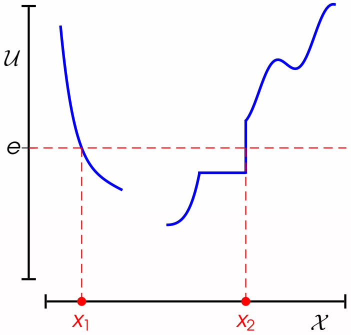

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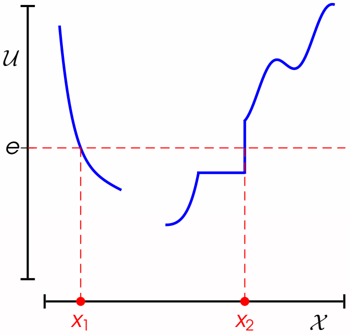

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- Clearly,

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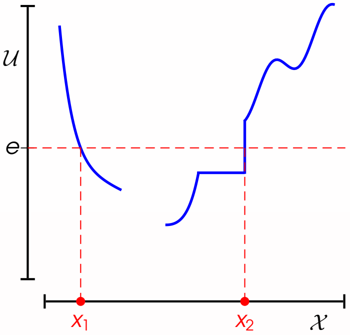

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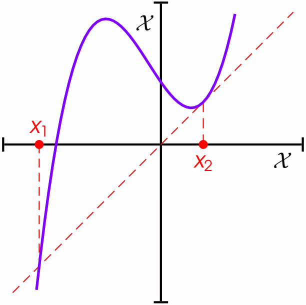

- Thus $Z = \operatorname{Fix} J_M^K$, where

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is the *warped resolvent* of M with *kernel* K.

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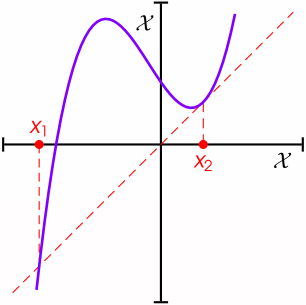

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- $p = J_M^K x \Leftrightarrow (p, Kx \boxminus Kp) \in \operatorname{gra} M$.

The warped resolvent

PART 2:

The warped resolvent

The warped resolvent: Definition

- $\mathcal{X}$ is a reflexive real Banach space with topological dual $\mathcal{X}^*$.
- An operator $M\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ is *monotone* if

$$(\forall (x_1, x_1^*) \in \operatorname{gra} M)\,(\forall (x_2, x_2^*) \in \operatorname{gra} M) \quad \langle x_1 - x_2, x_1^* - x_2^* \rangle \geqslant 0,$$

and *maximally monotone* if, furthermore, no point can be added to $\operatorname{gra} M$ without compromising monotonicity.

Definition

Let $\varnothing \neq D \subset \mathcal{X}$, let $K\colon D \to \mathcal{X}^*$, and let $M\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ be such that $\operatorname{ran} K \subset \operatorname{ran}(K + M)$ and $K + M$ is injective. The warped resolvent of M with kernel K is $J_M^K = (K + M)^{-1} \circ K$.

The warped resolvent: Properties

- Sufficient conditions for ran $K \subset \operatorname{ran}(K+M)$ and $K+M$ is injective are given in (Bùi/PLC, 2019).
- $J_M^K\colon D \to D$.
- $\operatorname{Fix} J_M^K = D \cap \operatorname{zer} M$.
- $p = J_M^K x \Leftrightarrow (p, Kx - Kp) \in \operatorname{gra} M$.
- Suppose that M is monotone. Let $x \in D$, and set $y = J_M^K x$ and $y^* = Kx - Ky$. Then

$$\operatorname{zer} M \subset \big\{ z \in \mathcal{X} \mid \langle z - y, y^* \rangle \leqslant 0 \big\}.$$

- Suppose that M is monotone. Set $p = J_M^K x$ and $q = J_M^K y$. Then

$$\langle p - q, Kx - Ky \rangle \geqslant \langle p - q, Kp - Kq \rangle.$$

The warped resolvent: Examples

$M\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ is maximally monotone.

- If $\mathcal{X}$ is Hilbertian and $K = \text{Id}$, J_M^K is the classical resolvent.

The warped resolvent: Examples

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- If $\mathcal{X}$ is Hilbertian and $K = \mathrm{Id}$, J_M^K is the classical resolvent.
- If $\mathcal{X}$ is strictly convex with normalized duality mapping K, then J_M^K is the extended resolvent of (Kassay, 1985).

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- Let $f\colon \mathcal{X} \to \left]-\infty, +\infty\right]$ be a Legendre function such that $\mathrm{dom}\, M \subset \mathrm{int}\,\mathrm{dom}\, f$, and set $K = \nabla f$. Then J_M^K is the D-resolvent of (Bauschke/Borwein/PLC, 2003).

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- $A\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ and $B\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ are maximally monotone, and $f\colon \mathcal{X} \to \left]-\infty, +\infty\right]$ is a suitable convex function. Set

$$M = A + B \quad \text{and} \quad K\colon \operatorname{int}\operatorname{dom} f \to \mathcal{X}^*\colon x \mapsto \nabla f(x) - Bx.$$

Then $J_M^K = (\nabla f + A)^{-1} \circ (\nabla f - B)$ is the Bregman forward-backward operator to be investigated in Part 4.

The warped resolvent: Examples

- Let $K\colon \mathcal{X} \to \mathcal{X}^*$ be strictly monotone, 3^* monotone, and surjective. Then J_M^K is the K-resolvent of (Bauschke/Wang/Yao, 2010).

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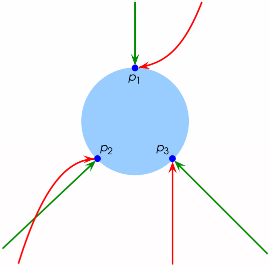
- Let $K\colon \mathcal{X} \to \mathcal{X}^*$ be strictly monotone, 3^* monotone, and surjective. Then J_M^K is the K-resolvent of (Bauschke/Wang/Yao, 2010).
- Let $\varnothing \neq C \subset \mathcal{X}$ be closed and convex, with normal cone operator N_C. The warped projection operator is $\operatorname{proj}_C^K = J_{N_C}^K = (K+N_C)^{-1}\circ K$.



Left: Warped projections onto $B(0;1)$. Sets of points projecting onto p_1, p_2, and p_3 for $K_1 = \mathrm{Id}$ and

$$K_2\colon (\xi_1,\xi_2) \mapsto \left(\frac{\xi_1^3}{2} + \frac{\xi_1}{5} - \xi_2, \xi_1 + \xi_2\right)$$

Note that K_2 is not a gradient.

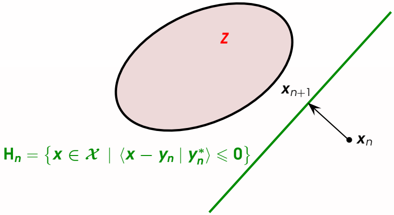
Warped proximal iterations in Hilbert space

PART 3:

Warped proximal iterations in Hilbert spaces

Finding zeros of monotone operators: Geometry

- M maximally monotone with $Z = \operatorname{zer} M \neq \varnothing$.



Finding zeros of monotone operators: Geometry

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- Iterate

$$
\left\lfloor
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\lambda_n \in [\varepsilon, 2-\varepsilon] \\
\text{if } \langle y_n - x_n \mid y_n^* \rangle < 0 \\
\left\lfloor\; x_{n+1} = x_n + \lambda_n \langle y_n - x_n \mid y_n^* \rangle y_n^* / \|y_n^*\|^2 \right. \\
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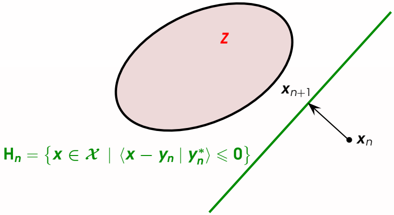

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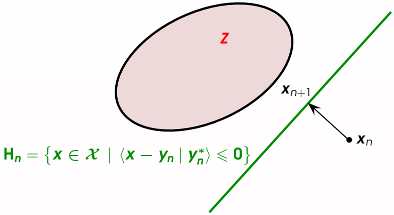
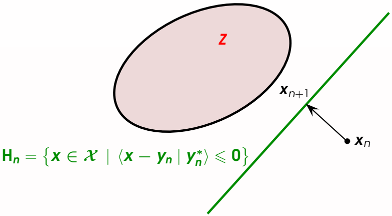


- Weak convergence to a point in Z if weak cluster points are in Z.
- The weak–to–strong convergence principle (Bauschke/PLC, 2001) gives strong convergence of a 2 half-spaces variant.
- How to choose $(y_n, y_n^*) \in \operatorname{gra} M$?

Finding zeros of monotone operators: Geometry

- M maximally monotone with $Z = \operatorname{zer} M \neq \varnothing$.
- Iterate

$$
\left\lfloor
\begin{array}{l}
y_n = J_{\gamma_n M}^{K_n} \widetilde{x}_n \\
y_n^* = \gamma_n^{-1}(K_n \widetilde{x}_n - K_n y_n) \\
\lambda_n \in [\varepsilon, 2-\varepsilon] \\
\text{if } \langle y_n - x_n \mid y_n^* \rangle < 0 \\
\left\lfloor \; x_{n+1} = x_n + \lambda_n \langle y_n - x_n \mid y_n^* \rangle y_n^* / \|y_n^*\|^2 \right. \\
\text{else} \\
\left\lfloor \; x_{n+1} = x_n. \right.
\end{array}
\right.
$$



- **Key:** Move beyond Minty's parametrization of $\operatorname{gra} M$ and use a warped resolvent to pick $(y_n, y_n^*) \in \operatorname{gra} M$.
- Simply evaluate a warped resolvent at some point $\widetilde{x}_n$.

Convergence

Notation: $(y^*)^\sharp = y^*/\|y^*\|$ if $y^* \neq 0$; $= 0$ otherwise.

Theorem

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, +\infty[$. For every $n \in \mathbb{N}$, let $\widetilde{x}_n \in \mathcal{X}$ and let $K_n \colon \mathcal{X} \to \mathcal{X}$ be a monotone operator such that $\operatorname{ran} K_n \subset \operatorname{ran}(K_n + \gamma_n M)$ and $K_n + \gamma_n M$ is injective. Suppose that:

- $\widetilde{x}_n - x_n \to 0$.
- $\left\langle \widetilde{x}_n - y_n \mid (K_n\widetilde{x}_n - K_n y_n)^\sharp \right\rangle \to 0 \quad \Rightarrow \quad \begin{cases} \widetilde{x}_n - y_n \rightharpoonup 0 \\ K_n\widetilde{x}_n - K_n y_n \to 0. \end{cases}$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in Z.

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Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in Z.

- We also have a strongly convergent version.

Choosing the evaluation points $(\widetilde{x}_n)_{n\in\mathbb{N}}$

The auxiliary sequence $(\widetilde{x}_n)_{n\in\mathbb{N}}$ can serve several purposes:

- $\widetilde{x}_n$ can model an additive perturbation of x_n, say $\widetilde{x}_n = x_n + e_n$, where we require only $\|e_n\| \to 0$.

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- More generally,

$$(\forall n \in \mathbb{N}) \quad \widetilde{x}_n = \sum_{j=0}^{n} \mu_{n,j} x_j.$$

with $\sum_{j=0}^{n} \mu_{n,j} = 1$ and $(1 - \mu_{n,n})x_n - \sum_{j=0}^{n-1} \mu_{n,j} x_j \to 0$.

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  with $\sum_{j=0}^{n} \mu_{n,j} = 1$ and $(1 - \mu_{n,n})x_n - \sum_{j=0}^{n-1} \mu_{n,j} x_j \to 0$.
- Nonlinear perturbations can also be considered. For instance, at iteration n, $\widetilde{x}_n = \mathrm{proj}_{C_n} x_n$ is an approximation to x_n from some suitable closed convex set $C_n \subset \mathcal{X}$.

Corollary 1

Corollary

Let $A\colon \mathcal{X} \to 2^{\mathcal{X}}$ *be maximally monotone, and let* $B\colon \mathcal{X} \to \mathcal{X}$ *be monotone and* β*-Lipschitzian, with* $\operatorname{zer}(A+B) \neq \varnothing$*. Let* $W_n\colon \mathcal{X} \to \mathcal{X}$ *be* α*-strongly monotone and* χ*-Lipschitzian, and let* $\gamma_n \in [\varepsilon, (\alpha-\varepsilon)/\beta]$*, let* $\lambda_n \in [\varepsilon, 2-\varepsilon]$*, and let* $\mathcal{X} \ni e_n \to 0$*. Furthermore, let* $m > 0$ *and let* $(\mu_{n,j})_{n\in\mathbb{N}, 0\leqslant j\leqslant n}$ *be bounded and satisfy*

- *For every* $n > m$ *and every integer* $j \in [0, n-m-1]$*,* $\mu_{n,j} = 0$*.*
- *For every* $n \in \mathbb{N}$*,* $\sum_{j=0}^{n} \mu_{n,j} = 1$*.*

Iterate

$$
\left\lfloor
\begin{array}{l}
\widetilde{x}_n = e_n + \sum_{j=0}^{n} \mu_{n,j} x_j \\
v_n^* = W_n\widetilde{x}_n - \gamma_n B\widetilde{x}_n \\
y_n = (W_n + \gamma_n A)^{-1} v_n^* \\
y_n^* = \gamma_n^{-1}(v_n^* - W_n y_n) + B y_n \\
\text{if } \langle y_n - x_n \mid y_n^* \rangle < 0 \\
\left\lfloor\; x_{n+1} = x_n + \dfrac{\lambda_n \langle y_n - x_n \mid y_n^* \rangle}{\|y_n^*\|^2}\, y_n^* \right. \\
\text{else } x_{n+1} = x_n.
\end{array}
\right.
$$

Then $(x_n)_{n\in\mathbb{N}}$ *converges weakly to a point in* $\operatorname{zer}(A+B)$*.*

Proof: $M = A + B$ and $K_n = W_n - \gamma_n B$. **Special case:** Tseng's algorithm.

Corollary 2: Multivariate inclusions

- **Problem:** find $(x_i)_{i\in I} \in \bigtimes_{i\in I} \mathcal{X}_i$ such that

$$(\forall i \in I) \quad 0 \in A_i x_i + \sum_{j\in J} L_{ji}^*\left((B_j + D_j)\left(\sum_{k\in I} L_{jk} x_k\right)\right) + C_i x_i$$

- **Warping:** Apply Theorem 2 to

$$M\colon \; ((x_i)_{i\in I}, (y_j)_{j\in J}, (v_j^*)_{j\in J}) \mapsto \left(\bigtimes_{i\in I}\left(A_i x_i + C_i x_i + \sum_{j\in J} L_{ji}^* v_j^*\right),\right.$$
$$\left.\bigtimes_{j\in J}(B_j y_j + D_j y_j - v_j^*), \bigtimes_{j\in J}\left\{y_j - \sum_{i\in I} L_{ji} x_i\right\}\right)$$

and $K_n\colon (x, y, v^*) \mapsto$

$$\left(\left(\gamma_{i,n}^{-1} F_{i,n} x_i - C_i x_i - \sum_{j\in J} L_{ji}^* v_j^*\right)_{i\in I}, (\tau_{j,n}^{-1} W_{j,n} y_j - D_j y_j + v_j^*)_{j\in J},\right.$$
$$\left.\left(-y_j + v_j^* + \sum_{i\in I} L_{ji} x_i\right)_{j\in J}\right),$$

where $F_{i,n}$ and $W_{j,n}$ are strongly monotone and Lipschitzian.

Corollary 2: Multivariate inclusions

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\quad \text{for every } i \in I \\
\quad\quad l_{i,n}^* = F_{i,n}\widetilde{x}_{i,n} - \gamma_{i,n} C_i \widetilde{x}_{i,n} - \gamma_{i,n} \sum_{j \in J} L_{ji}^* \widetilde{v}_{j,n}^* \\
\quad\quad a_{i,n} = \big(F_{i,n} + \gamma_{i,n} A_i\big)^{-1} (l_{i,n}^* + \gamma_{i,n} s_i^*) \\
\quad\quad a_{i,n}^* = \gamma_{i,n}^{-1} (l_{i,n}^* - F_{i,n} a_{i,n}) + C_i a_{i,n} \\
\quad \text{for every } j \in J \\
\quad\quad t_{j,n}^* = W_{j,n} \widetilde{y}_{j,n} - \tau_{j,n} D_j \widetilde{y}_{j,n} + \tau_{j,n} \widetilde{v}_{j,n}^* \\
\quad\quad b_{j,n} = \big(W_{j,n} + \tau_{j,n} B_j\big)^{-1} t_{j,n}^* \\
\quad\quad f_{j,n}^* = \tau_{j,n}^{-1} (t_{j,n}^* - W_{j,n} b_{j,n}) + D_j b_{j,n} \\
\quad\quad c_{j,n} = \sum_{i \in I} L_{ji} \widetilde{x}_{i,n} - \widetilde{y}_{j,n} + \widetilde{v}_{j,n}^* - r_j \\
\quad \text{for every } i \in I \\
\quad\quad a_{i,n}^* = o_{i,n}^* + \sum_{j \in J} L_{ji}^* c_{j,n} \\
\quad \text{for every } j \in J \\
\quad\quad b_{j,n}^* = f_{j,n}^* - c_{j,n} \\
\quad\quad c_{j,n}^* = r_j + b_{j,n} - \sum_{i \in I} L_{ji} a_{i,n} \\
\quad \sigma_n = \sum_{i \in I} \|a_{i,n}^*\|^2 + \sum_{j \in J} \big(\|b_{j,n}^*\|^2 + \|c_{j,n}^*\|^2\big) \\
\quad \theta_n = \sum_{i \in I} \langle a_{i,n} - x_{i,n} \mid a_{i,n}^* \rangle + \sum_{j \in J} \big(\langle b_{j,n} - y_{j,n} \mid b_{j,n}^* \rangle + \langle c_{j,n} - v_{j,n}^* \mid c_{j,n}^* \rangle\big) \\
\quad \text{if } \theta_n < 0 \\
\quad\quad \rho_n = \lambda_n \theta_n / \sigma_n \\
\quad \text{else} \\
\quad\quad \rho_n = 0 \\
\quad \text{for every } i \in I \\
\quad\quad x_{i,n+1} = x_{i,n} + \rho_n a_{i,n}^* \\
\quad \text{for every } j \in J \\
\quad\quad y_{j,n+1} = y_{j,n} + \rho_n b_{j,n}^* \\
\quad\quad v_{j,n+1}^* = v_{j,n}^* + \rho_n c_{j,n}^*.
\end{array}
$$

Further connections

- Primal-dual splitting.
  - Consider the inclusion $0 \in Ax + L^*(B(Lx))$ and the associated Kuhn–Tucker operator

$$M\colon \mathcal{X} \times \mathcal{Y} \to 2^{\mathcal{X} \times \mathcal{Y}} \colon (x, y^*) \mapsto (Ax + L^* y^*) \times (-Lx + B^{-1} y^*).$$

  - The cutting plane method of (Alotaibi/PLC/Shahzad, 2014) and (PLC/Eckstein, 2018) generate points $(a_n, a_n^*) \in \operatorname{gra} A$ and $(b_n, b_n^*) \in \operatorname{gra} B$. This implicitly provides

$$(y_n, y_n^*) = \big((a_n, b_n^*), (a_n^* + L^* b_n^*, -La_n + b_n)\big) \in \operatorname{gra} M$$

    to construct $H_n \supset \operatorname{zer} M$.
  - The primal-dual framework of (Alotaibi/PLC/Shahzad, 2014) is therefore an instance of Theorem 2 with

$$K_n\colon (x, y^*) \mapsto \big(\gamma_n^{-1} x - L^* y^*, Lx + \mu_n y^*\big).$$

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- An alternate cutting plane strategy was independently investigated in (Giselsson, arXiv 2019), where an instance of a warped resolvent (in our sense) was used.

Warped proximal iterations with Bregman kernels

PART 4:

Warped proximal iterations with Bregman kernels

Bregman forward-backward splitting

- $\mathcal{X}$ a reflexive real Banach space, $A\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ and $B\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ maximally monotone, and $f \in \Gamma_0(\mathcal{X})$ essentially smooth.
- $C = (\operatorname{int}\operatorname{dom} f) \cap \operatorname{dom} A \subset \operatorname{int}\operatorname{dom} B$ and B is single-valued on $\operatorname{int}\operatorname{dom} B$.
- $(\forall x \in C)(\forall y \in C)(\forall z \in \mathscr{S})(\forall y^* \in Ay)(\forall z^* \in Az)$
$$\langle y - x, By - Bz\rangle \leqslant \kappa D_f(x, y) + \big\langle y - z, \delta_1(y^* - z^*) + \delta_2\big(By - Bz\big)\big\rangle.$$
- The objective is to
$$\text{find } x \in \mathscr{S} = (\operatorname{int}\operatorname{dom} f) \cap \operatorname{zer}(A + B) \neq \varnothing.$$

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- The objective is to
$$\text{find } x \in \mathscr{S} = (\operatorname{int}\operatorname{dom} f) \cap \operatorname{zer}(A + B) \neq \varnothing.$$
- Apply the warped proximal point algorithm
$$x_{n+1} = J_M^{K_n} x_n$$
to $M = A + B$ with kernel $K_n = \gamma_n^{-1}\nabla f_n - B$ for a suitable essentially smooth function f_n.

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to $M = A + B$ with kernel $K_n = \gamma_n^{-1}\nabla f_n - B$ for a suitable essentially smooth function f_n.
- We obtain the Bregman forward-backward splitting algorithm
$$x_{n+1} = \big(\nabla f_n + \gamma_n A\big)^{-1}\big(\nabla f_n(x_n) - \gamma_n B x_n\big).$$

Convergence

Theorem

"Under suitable assumptions,"

$$x_{n+1} = \big(\nabla f_n + \gamma_n A\big)^{-1}\big(\nabla f_n(x_n) - \gamma_n B x_n\big) \rightharpoonup x \in \mathscr{S}.$$

- This result provides, for instance, the convergence of the basic Bregman forward-backward splitting method

$$\big(\nabla f + \gamma A\big)^{-1}\big(\nabla f(x_n) - \gamma B x_n\big),$$

  which is new even in Euclidean spaces.
- It also allows us to recover and extend 4, so far unrelated, splitting frameworks.

$x_{n+1} = (\nabla f_n + \gamma_n A)^{-1}(\nabla f_n(x_n) - \gamma_n B x_n)$: Instantiations

- The iteration $x_{n+1} = \big(\nabla f + \gamma_n A\big)^{-1}\big(\nabla f(x_n)\big)$ for finding a zero of A in a reflexive Banach space (Bauschke/Borwein/PLC, 2003).
- The iteration $x_{n+1} = \big(U_n + \gamma_n A\big)^{-1}\big(U_n x_n - \gamma_n B x_n\big)$ for finding a zero of $A + B$ in a Hilbert space, where U_n is a strongly positive Hermitian bounded linear operator (PLC/Vũ, 2014).
- The iteration

$$x_{n+1} = \big(\nabla f + \gamma A\big)^{-1}\big(\nabla f(x_n) - \gamma B x_n\big)$$

  for finding a zero of $A + B$ in a Hilbert space, where f is real-valued and strongly convex (Renaud/Cohen, 1997).
- The iteration

$$x_{n+1} = \big(\nabla f_n + \gamma_n \partial\varphi\big)^{-1}\big(\nabla f_n(x_n) - \gamma_n \nabla\psi(x_n)\big)$$

  for minimizing $\varphi + \psi$ in a reflexive Banach space (Nguyen, 2017; see also Bauschke/Bolte/Teboulle, 2017).

Illustration: The minimization setting

Let $\varphi \in \Gamma_0(\mathcal{X})$, $\psi \in \Gamma_0(\mathcal{X})$, and $f \in \Gamma_0(\mathcal{X})$ be essentially smooth. Set $C = (\operatorname{int}\operatorname{dom} f) \cap \operatorname{dom}\partial\varphi$ and $\mathscr{S} = (\operatorname{int}\operatorname{dom} f) \cap \operatorname{Argmin}(\varphi+\psi)$. Suppose that $C \neq \varnothing$, $\varphi+\psi$ is coercive, $C \subset \operatorname{int}\operatorname{dom}\psi$, $\mathscr{S} \neq \varnothing$, ψ is Gâteaux differentiable on $\operatorname{int}\operatorname{dom}\psi$, and $D_f \geqslant \beta D_\psi$.

Corollary

Take $x_0 \in C$ and set

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = (\nabla f_n + \gamma_n \partial\varphi)^{-1}(\nabla f_n(x_n) - \gamma_n \nabla\psi(x_n)).$$

Then:

- $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in $\mathscr{S}$.
- $(\varphi+\psi)(x_n) - \min(\varphi+\psi)(\mathcal{X}) = o(1/n)$.
- $\sum_{n\in\mathbb{N}} n(D_{f_n}(x_{n+1}, x_n) + D_{f_n}(x_n, x_{n+1})) < +\infty$.

- Weak convergence was obtained in (Nguyen, 2017) under more restrictive assumptions.
- The rates are new, even in Euclidean spaces.

References

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- PLC, Monotone operator theory in convex optimization, *Math. Programming*, 2018.
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Bregman distance

- $f \in \Gamma_0(\mathcal{X})$ is a Legendre function if it is both (Bauschke/Borwein/PLC, 2001):
  - Essentially smooth: ∂f is both locally bounded and single-valued on its domain.
  - Essentially strictly convex: ∂f^* is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} \partial f$.
- Take $f \in \Gamma_0(\mathcal{X})$, Gâteaux differentiable on $\operatorname{int} \operatorname{dom} f \neq \varnothing$. The associated Bregman distance is

$$D_f \colon \mathcal{X} \times \mathcal{X} \to [0, +\infty]$$
$$(x, y) \mapsto \begin{cases} f(x) - f(y) - \langle x - y, \nabla f(y) \rangle, & \text{if } y \in \operatorname{int} \operatorname{dom} f; \\ +\infty, & \text{otherwise.} \end{cases}$$