GAMES, DYNAMICS & OPTIMIZATION

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One World Optimization / Game Theory Seminar – July 13, 2020

Outline

Overview

From flows to algorithms

From algorithms to flows

Flows in games

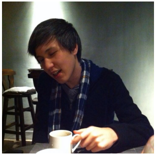
Monotone games

Spurious limits

About

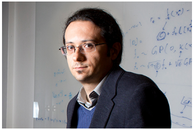
N. Hallak

A. Kavis



Y. -P. Hsieh

C. Papadimitriou



V. Cevher

G. Piliouras

Z. Zhou

- M, Papadimitriou & Piliouras, *Cycles in adversarial regularized learning,* SODA 2018
- M & Zhou, *Learning in games with continuous action sets and unknown payoff functions,* Mathematical Programming, vol. 173, pp. 465–507, Jan. 2019
- M, Hallak, Kavis & Cevher, *On the almost sure convergence of stochastic gradient descent in non-convex problems,* `https://arxiv.org/abs/2006.11144`
- Hsieh, M & Cevher, *The limits of min-max optimization algorithms: convergence to spurious non-critical sets,* `https://arxiv.org/abs/2006.09065`

Background

The use of dynamical systems in optimization & game theory

Dynamics and optimization

- Gradient flows [Too many to list]
- Non-Euclidean flows [Alvarez, Bolte, Bomze, Jordan, Teboulle, Wibisono,…]
- Heavy ball methods [Alvarez, Attouch, Bolte, Cominetti, Goudou, Redont,…]
- Accelerated methods [Attouch, Boyd, Candès, Peypouquet, Su,…]
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Dynamics and games

- Population dynamics [Sandholm,...]
- Random matching [Hofbauer, Sigmund, Weibull,...]
- Learning in games [Benaïm, Hart, Hofbauer, Mas-Colell, Sorin,...]
- ...

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Basic problem

$$\text{minimize}_{x \in \mathbb{R}^d} \quad f(x)$$

- f non-convex [technical assumptions later]
- f unknown/difficult to manipulate in closed form [low precision methods]
- Single-player game: calculate best responses [more in second part]

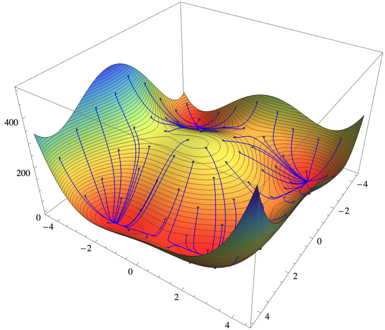
Gradient flows

The *gradient flow* of a function $f\colon \mathbb{R}^d \to \mathbb{R}$

$$\dot{x}(t) = -\nabla f(x(t)) \tag{GF}$$

Main property: f is a (strict) *Lyapunov function* for (GF)

$$df/dt = -\|\nabla f(x(t))\|^2 \leq 0 \qquad \text{w/ equality iff } \nabla f(x) = 0$$



Convergence of gradient flows

Blanket assumptions

- *Lipschitz smoothness:*

$$\|\nabla f(x') - \nabla f(x)\| \leq L\|x' - x\| \quad \text{for all } x, x' \in \mathbb{R}^d \tag{LS}$$

- *Bounded sublevels:*

$$L_c \equiv \{x \in \mathbb{R}^d : f(x) \leq c\} \quad \text{is bounded for all } c < \sup f \tag{Bsub}$$

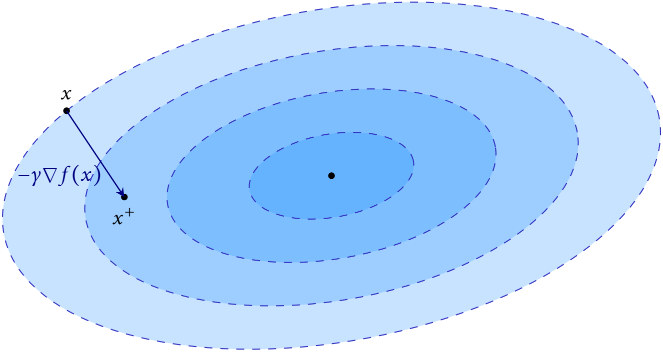
Theorem

- Assume: (LS), (Bsub)
- Then: $x(t)$ *converges to* $\operatorname{crit}(f) \equiv \{x^* \in \mathbb{R}^d : \nabla f(x^*) = 0\}$

From flows to algorithms: gradient descent

Forward Euler (explicit) $\Longrightarrow$ gradient descent (GD) [Cauchy, 1847]

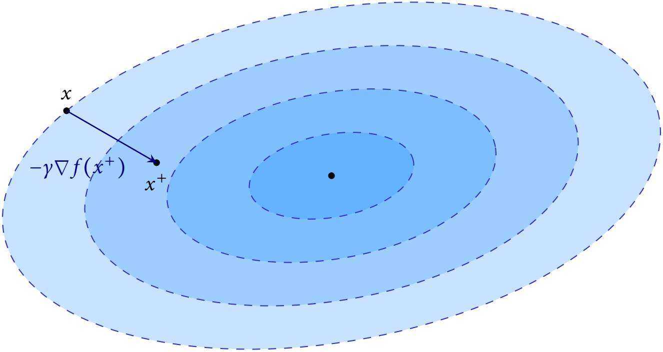
$$X_{n+1} = X_n - \gamma_n \nabla f(X_n) \tag{GD}$$



From flows to algorithms: proximal gradient

Backward Euler (implicit) $\Longrightarrow$ proximal gradient (PG) [Martinet, 1970]

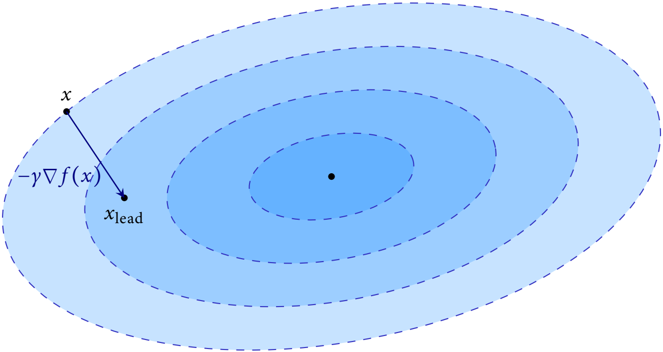
$$X_{n+1} = X_n - \gamma_n \nabla f(X_{n+1}) \tag{PG}$$



From flows to algorithms: extra-gradient

Midpoint Runge-Kutta (explicit) $\Longrightarrow$ **extra-gradient (EG)** [Korpelevich, 1976]

$$X_{n+1/2} = X_n - \gamma_n \nabla f(X_n) \qquad X_{n+1} = X_n - \gamma_n \nabla f(X_{n+1/2}) \tag{EG}$$



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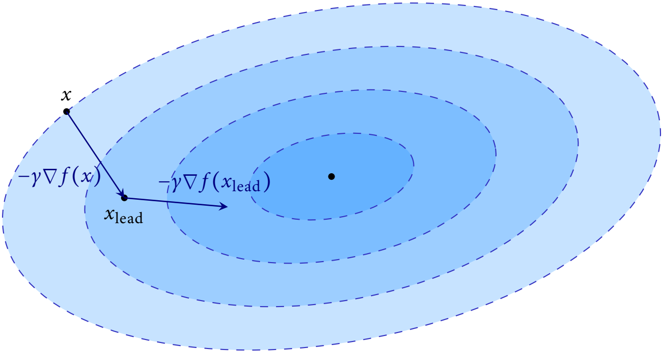

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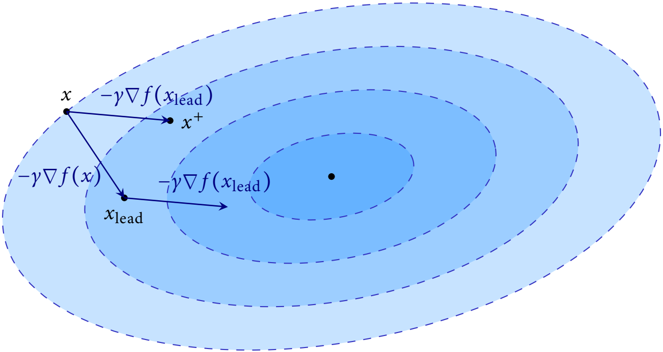

Stochastic gradient feedback

In many applications, perfect gradient information is unavailable / too costly:

- **Machine learning:**
  $f(x) = \sum_{i=1}^{N} f_i(x)$ and only a batch of $\nabla f_i(x)$ is computable per iteration

- **Control / Engineering:**
  $f(x) = \mathbb{E}[F(x;\omega)]$ and only $\nabla F(x;\omega)$ can be observed for a random ω

- **Game Theory / Bandit Learning:**
  Only $f(x)$ is observable

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Stochastic first-order oracle (SFO) feedback:

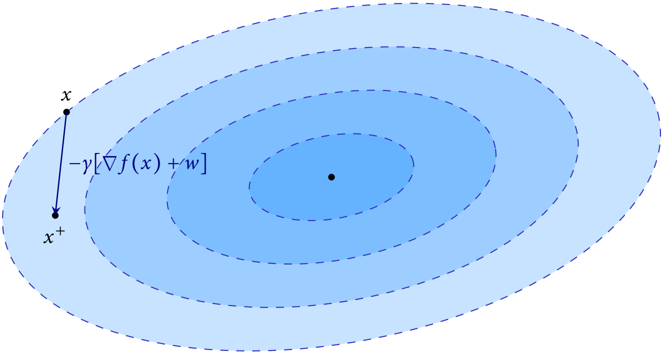
$$X_n \mapsto \underbrace{V_n}_{\text{feedback}} = \underbrace{\nabla f(X_n)}_{\text{gradient}} + \underbrace{Z_n}_{\text{noise}} + \underbrace{b_n}_{\text{bias}} \tag{SFO}$$

where Z_n is "zero-mean" and b_n is "small" (more later)

Stochastic gradient descent

Noisy Euler (explicit) $\Longrightarrow$ stochastic gradient descent (SGD)

$$X_{n+1} = X_n - \gamma_n [\nabla f(X_n) + \underbrace{W_n}_{\text{noise}}] \tag{SGD}$$



Example: zeroth-order feedback

Given $f\colon \mathbb{R} \to \mathbb{R}$, estimate $f'(x)$ at target point $x \in \mathbb{R}$

$$f'(x) \approx \frac{f(x+\delta) - f(x-\delta)}{2\delta}$$

Pick $u = \pm 1$ with probability $1/2$. Then:

$$\mathbb{E}[f(x+\delta u)u] = \frac{1}{2} f(x+\delta) - \frac{1}{2} f(x-\delta)$$

$\implies$ Estimate $f'(x)$ with a single query of f at $\hat{x} = x + \delta u$

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Algorithm 1 Simultaneous perturbation stochastic approximation [Spall, 1992]

1: Draw u uniformly from $\mathbb{S}^d$
2: **Query** $\hat{x} = x + \delta u$
3: **Get** $\hat{f} = f(\hat{x})$
4: **Set** $V = (d/\delta)\hat{f} u$

The Robbins-Monro template

Special cases of the generalized Robbins-Monro scheme

$$X_{n+1} = X_n - \gamma_n[\nabla f(X_n) + Z_n + b_n] \tag{RM}$$

with $\sum_n \gamma_n = \infty$, $\gamma_n \to 0$, and $\mathbb{E}[Z_n \mid X_n, \dots, X_1] = 0$

Examples

- Gradient descent (det.): $Z_n = 0,\ b_n = 0$
- Proximal gradient (det.): $Z_n = 0,\ b_n = \nabla f(X_{n+1}) - \nabla f(X_n)$
- Extra-gradient (det.): $Z_n = 0,\ b_n = \nabla f(X_{n+1/2}) - \nabla f(X_n)$
- Stochastic gradient descent (stoch.): $Z_n =$ zero-mean, $b_n = 0$
- SPSA (stoch.): $Z_n = (d/\delta) f(\hat{X}_n) U_n - \nabla f_\delta(X_n),\ b_n = \nabla f_\delta(X_n) - \nabla f(X_n)$ where

$$f_\delta(x) = \frac{1}{\mathrm{vol}(\mathbb{B}_\delta)} \int_{\mathbb{B}_\delta} f(x + \delta u)\, du$$

- ...

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Basic idea: *if γ_n is "small", the noise washes out and "$\lim_{t\to\infty}$ (RM) = $\lim_{t\to\infty}$ (GF)"*

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Basic idea: *if γ_n is "small", the noise washes out and "$\lim_{t\to\infty}$ (RM) = $\lim_{t\to\infty}$ (GF)"*

$\implies$ **ODE method of stochastic approximation**

[Ljung, 1977; Benveniste et al, 1990; Duflo, 1996; Kushner & Yin, 1997; Benaïm, 1999; …]

- Time interpolation: $\tau_n = \sum_{k=1}^{n} \gamma_k$
- Trajectory interpolation: $X(t) = X_n + \dfrac{t - \tau_n}{\tau_{n+1} - \tau_n}(X_{n+1} - X_n)$
- X_n is an **asymptotic pseudotrajectory (APT)** of (GF) if, for all $T > 0$:

$$\lim_{t\to\infty} \sup_{0\leq h\leq T} \| X(t+h) - \Phi_h(X(t)) \| = 0$$

  where $\Phi_s(x)$ denotes the position at time s of an orbit of (GF) starting at x
- **Long run:** $X(t)$ tracks (GF) with arbitrary accuracy over windows of arbitrary length

[Benaïm & Hirsch, 1995, 1996; Benaïm, 1999; Benaïm, Hofbauer & Sorin, 2005, 2006;…]

Stochastic approximation criteria

When is a sequence generated by (RM) an APT?

(A)
- X_n is bounded
- f is *Lipschitz continuous and smooth:*

$$|f(x') - f(x)| \leq G\|x' - x\| \tag{LC}$$

$$\|\nabla f(x') - \nabla f(x)\| \leq L\|x' - x\| \tag{LS}$$

(B)
- $\mathbb{E}[\sum_n \gamma_n^2 \|Z_n\|^2] < \infty$
- $\sup_n \mathbb{E}[\|Z_n\|^q] < \infty$ and $\sum_n \gamma_n^{1+q/2} < \infty$
- Z_n sub-Gaussian and $\gamma_n = o(1/\log n)$

(C)
- $\sum_n \gamma_n b_n = 0$ with probability 1

Proposition (Duflo 1996; Benaïm 1999; Hsieh, M & Cevher, 2020)

- **Assume:** *any of* **(A)**; *any of* **(B)**; **(C)**
- **Then:** X_n *is an APT of* (GF) *with probability* 1

Convergence of APTs

Theorem (Benaïm & Hirsch, 1995, 1996)

- **Assume:** X_n *is a bounded APT of* (GF)
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Theorem (Ljung 1977, Benaïm 1999)

- **Assume:** (LC), (LS), (Bsub); $\sup_n \|X_n\| < \infty$
- **Then:** X_n *converges* (a.s.) *to a component of* $\operatorname{crit}(f)$ *where* f *is constant*

Boundedness: implicit, algorithm-dependent assumption; not easy to verify

Can boundedness be dropped?

Key obstacle: infinite plains of vanishing gradients [think $f(x) = -\exp(-x^2)$]

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Countered if *gradient sublevel sets* do not extend to infinity

$$M_\varepsilon \equiv \{x \in \mathbb{R}^d : \|\nabla f(x)\| \leq \varepsilon\} \quad \text{is bounded for some } \varepsilon > 0 \tag{Gsub}$$

[$\implies \operatorname{crit}(f)$ compact]

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Proposition (M, Hallak, Kavis & Cevher, 2020)

- **Assume:** (LC), (LS), (Bsub), (Gsub)
- **Then:** *for all $\varepsilon > 0$, there exists some $\tau = \tau(\varepsilon)$ such that, for all $t \geq \tau$:*
  (a) $f(x(t)) \leq f(x(0)) - \varepsilon$*; or*
  (b) $x(t)$ *is within ε-distance of* $\operatorname{crit}(f)$

In words: (GF) either descends f by a uniform amount, or it is already near-critical

Can boundedness be dropped?

Proposition

- Assume: (LC), (LS), (Bsub), (Gsub); *any of* (B); (C)
- Then: *with probability* 1*, there exists a (random) subsequence* X_{n_k} *of* X_n *converging to a critical point of* f

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Theorem (M, Hallak, Kavis & Cevher, 2020)

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Are all critical points desirable?

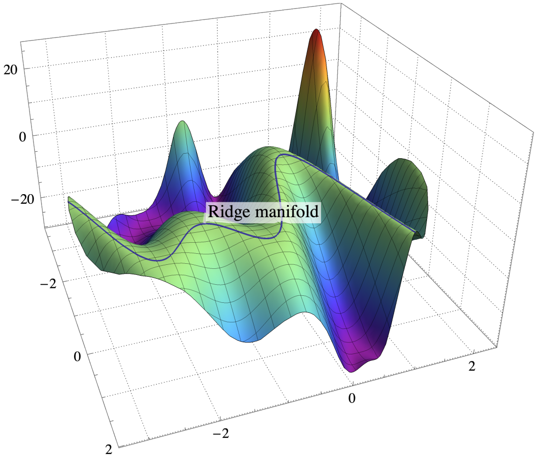

Figure: A hyperbolic ridge manifold, typical of ResNet loss landscapes [Li et al., 2018]

Are traps avoided?

Hyperbolic saddle (isolated non-minimizing critical point)

$$\lambda_{\min}(\mathrm{Hess}(f(x^*))) < 0, \quad \det(\mathrm{Hess}(f(x^*))) \neq 0$$

$\Longrightarrow$ (GF) is linearly unstable near x^*

$\Longrightarrow$ convergence to x^* unlikely

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Theorem (Pemantle, 1990)

- Assume:
  - x^* *is a hyperbolic saddle point*
  - Z_n *is finite* (a.s.) *and uniformly exciting*

$$\mathbb{E}[\langle Z, u\rangle^+] \geq c \quad \text{for all unit vectors } u \in \mathbb{S}^{d-1},\ x \in \mathbb{R}^d$$

  - $\gamma_n \propto 1/n$
- Then: $\mathbb{P}(\lim_{n\to\infty} X_n = x^*) = 0$

Are non-hyperbolic traps avoided?

Strict saddle

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Theorem (Ge, Huang, Jin & Yuan, 2015)

- **Given:** *confidence level* $\zeta > 0$
- **Assume:**
  - f *is bounded and satisfies* (LS)
  - $\operatorname{Hess}(f(x))$ *is Lipschitz continuous*
  - *for all* $x \in \mathbb{R}^d$: (a) $\|\nabla f(x)\| \geq \varepsilon$; *or* (b) $\lambda_{\min}(\operatorname{Hess}(f(x))) \leq -\beta$; *or* (c) x *is* δ*-close to a local minimum* x^* *of* f *around which* f *is* α*-strongly convex*
  - Z_n *is finite* (a.s.) *and contains a component uniformly sampled from the unit sphere; also,* $b_n = 0$
  - $\gamma_n \equiv \gamma$ *with* $\gamma = \mathcal{O}(1/\log(1/\zeta))$
- **Then:** *with probability at least* $1-\zeta$*, the algorithm produces after* $\mathcal{O}(\gamma^{-2}\log(1/(\gamma\zeta)))$ *iterations a point which is* $\mathcal{O}(\sqrt{\gamma}\log(1/(\gamma\zeta)))$*-close to* x^* *(and hence away from any strict saddle)*

*Are non-hyperbolic traps avoided **always**?*

Theorem (M, Hallak, Kavis & Cevher, 2020)

- **Assume:**
  - f *satisfies* (LC) *and* (LS)
  - Z_n *is finite* (a.s.) *and* *uniformly exciting*

$$\mathbb{E}[\langle Z, u\rangle^+] \geq c \quad \text{for all unit vectors } u \in \mathbb{S}^{d-1},\ x \in \mathbb{R}^d$$

  - $\gamma_n \propto 1/n^p$ *for some* $p \in (0,1]$
- **Then:** $\mathbb{P}(X_n \text{ converges to a set of strict saddle points}) = 0$

Proof.

Use Pemantle (1990) + differential geometric arguments of Benaïm and Hirsch (1995). □

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Single- vs. multi-agent setting

In single-agent optimization, first-order iterative schemes

- Converge to the problem's set of critical points
- Avoid spurious, non-minimizing critical manifolds

Single- vs. multi-agent setting

In **single-agent optimization**, first-order iterative schemes

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Does this intuition carry over to games?

Do **multi-agent** learning algorithms

- Converge to unilaterally stable/stationary points?
- Avoid spurious, non-equilibrium points?

Online decision processes

Agents called to take repeated decisions with minimal information:

---

for $n \geq 0$ **do**

  Choose action X_n [focal agent choice]

  Incur loss $f_n(X_n)$ [depends on all agents]

end for

Driving question: *How to choose "good" actions?*

- Unknown world: no beliefs, knowledge of the game, etc.
- Minimal information: feedback often limited to incurred losses

N-player games

The game

- Finite set of **players** $i \in \mathcal{N} = \{1, \dots, N\}$
- Each player selects an **action** from a closed convex set $\mathcal{X}_i \subseteq \mathbb{R}^{d_i}$
- Loss of player i given by **cost function** $f_i \colon \mathcal{X} \equiv \prod_i \mathcal{X}_i \to \mathbb{R}$

Examples

- Finite games (mixed extensions)
- Divisible good auctions (Kelly)
- Traffic routing
- Power control/allocation problems
- Cournot oligopolies
- ...

Nash equilibrium

Nash equilibrium

Action profile $x^* = (x_1^*, \dots, x_n^*) \in \mathcal{X}$ that is **unilaterally stable**

$$f_i(x_i^*; x_{-i}^*) \leq f_i(x_i; x_{-i}^*) \quad \text{for every player } i \in \mathcal{N} \text{ and every deviation } x_i \in \mathcal{X}_i$$

- Local version: **local Nash equilibrium** [stable under local deviations]
- Unilateral stationarity: **critical points** of the game [x_i^* is stationary for $f_i(\cdot, x_{-i}^*)$]

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Individual loss gradients

$$V_i(x) = \nabla_{x_i} f_i(x_i; x_{-i})$$

$\implies$ direction of individually steepest descent

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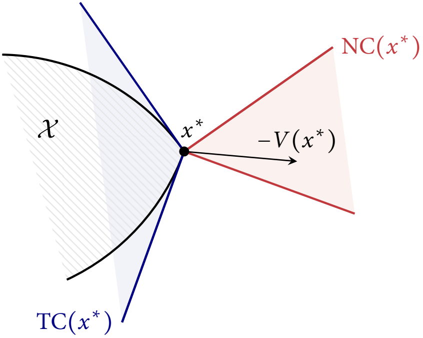
Variational characterization

If x^* is a (local) Nash equilibrium, then

$$\langle V_i(x^*), x_i - x_i^* \rangle \geq 0 \quad \text{for all } i \in \mathcal{N}, x_i \in \mathcal{X}_i$$

Intuition: $f_i(x_i; x_{-i}^*)$ weakly increasing along all rays emanating from x_i^*

Geometric interpretation



At Nash equilibrium, individual descent directions are outward-pointing

First-order algorithms in games

Individual gradient field $V(x) = (V_1(x), \dots, V_N(x))$, $x = (x_1, \dots, x_N)$

- **Individual gradient descent:**

$$X_{n+1} = X_n - \gamma_n V(X_n)$$

- **Extra-gradient:**

$$X_{n+1/2} = X_n - \gamma_n \nabla f(X_n) \qquad X_{n+1} = X_n - \gamma_n \nabla f(X_{n+1/2})$$

- ...

Mean dynamics:

$$\dot{x}(t) = -V(x(t)) \tag{MD}$$

$\Longrightarrow$ no longer a gradient system

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The dynamics of min-max games

Bilinear min-max games (saddle-point problems)

$$\min_{x_1 \in \mathcal{X}_1} \max_{x_2 \in \mathcal{X}_2} \quad L(x_1, x_2) = (x_1 - b_1)^\top A(x_2 - b_2) \tag{SP}$$

[no constraints: $\mathcal{X}_1 = \mathbb{R}^{d_1}$, $\mathcal{X}_2 = \mathbb{R}^{d_2}$]

Mean dynamics:

$$\dot{x}_1 = -A(x_2 - b_2) \qquad \dot{x}_2 = A^\top(x_1 - b_1)$$

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Bilinear min-max games (saddle-point problems)

$$\min_{x_1 \in \mathcal{X}_1} \max_{x_2 \in \mathcal{X}_2} \quad L(x_1, x_2) = (x_1 - b_1)^\top A (x_2 - b_2) \tag{SP}$$

[no constraints: $\mathcal{X}_1 = \mathbb{R}^{d_1}$, $\mathcal{X}_2 = \mathbb{R}^{d_2}$]

Mean dynamics:

$$\dot{x}_1 = -A(x_2 - b_2) \qquad \dot{x}_2 = A^\top (x_1 - b_1)$$

Energy function:

$$E(x) = \frac{1}{2}\|x_1 - b_1\|^2 + \frac{1}{2}\|x_2 - b_2\|^2$$

Lyapunov property:

$$\frac{dE}{dt} \leq 0 \quad \text{w/ equality if } A = A^\top$$

$\implies$ distance to solutions (weakly) decreasing along (MD)

Cycles

Roadblock: the energy might be a constant of motion [Hofbauer et al, 2009]

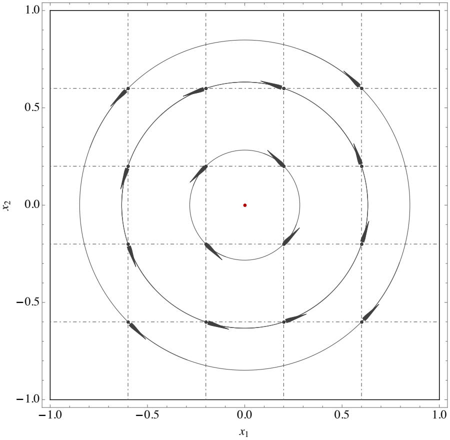
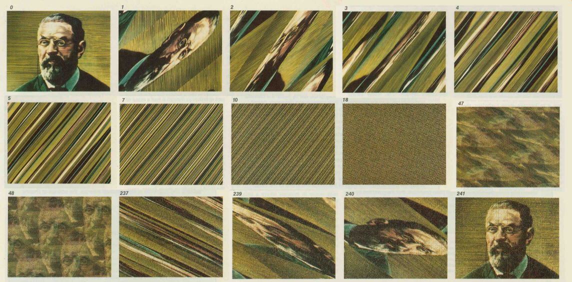


Figure: Hamiltonian flow of $L(x_1, x_2) = x_1 x_2$

Poincaré recurrence

Definition (Poincaré, 1890's)

A dynamical system is **Poincaré recurrent** if almost all solution trajectories return *arbitrarily close* to their starting point *infinitely many times*



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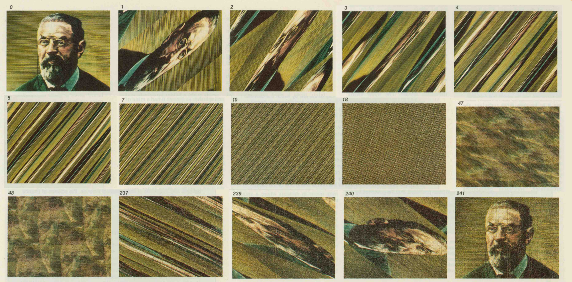

Theorem (M, Papadimitriou, Piliouras, 2018; unconstrained version)

(MD) *is Poincaré recurrent in all bilinear min-max games that admit an equilibrium*

Learning in min-max games: gradient descent

Individual gradient descent:

$$X_{n+1} = X_n - \gamma_n V(X_n)$$

Learning in min-max games: gradient descent

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Energy no longer a constant:

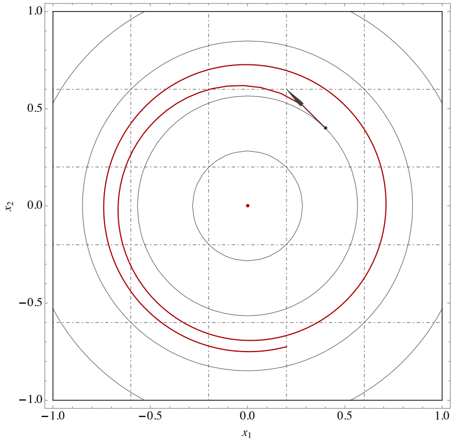
$$\frac{1}{2}\|X_{n+1} - x^*\|^2 = \frac{1}{2}\|X_n - x^*\|^2 + \gamma_n \underbrace{\cancel{\langle V(X_n), X_n - x^* \rangle}}_{\text{from (MD)}} + \frac{1}{2}\underbrace{\gamma_n^2 \|V(X_n)\|^2}_{\text{discretization error}}$$

...even worse

Learning in min-max games: gradient descent

Individual gradient descent:

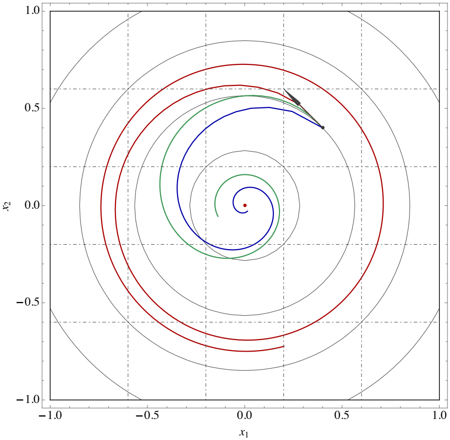
$$X_{n+1} = X_n - \gamma_n V(X_n)$$



Learning in min-max games: extra-gradient

Extra-gradient:

$$X_{n+1/2} = X_n - \gamma_n \nabla f(X_n) \qquad X_{n+1} = X_n - \gamma_n \nabla f(X_{n+1/2})$$



Learning in min-max games

Long-run behavior of min-max learning algorithms:

- **Mean dynamics:** Poincaré recurrent (periodic orbits)
- **Individual gradient descent:** divergence (outward spirals)
- **Extra-gradient:** convergence (inward spirals)

Learning in min-max games

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Different outcomes despite same underlying dynamics!

Monotonicity and strict monotonicity

Bilinear games are special cases of **monotone games:**

$$\langle V(x') - V(x), x' - x \rangle \geq 0 \quad \text{for all } x, x' \in \mathcal{X} \tag{MC}$$

[$\Longrightarrow$ **strictly monotone** if (MC) is strict for $x \neq x'$]

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Equivalently: $H(x) \succcurlyeq 0$ where H is the game's **Hessian matrix:**

$$H_{ij}(x) = \frac{1}{2}\nabla_{x_j}\nabla_{x_j} f_i(x) + \frac{1}{2}(\nabla_{x_i}\nabla_{x_j} f_j(x))^\top$$

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Examples: bilinear games (not strict), Kelly auctions, Cournot markets, routing, …

Nomenclature:

- **Diagonal strict convexity** [Rosen, 1965]
- Stable games [Hofbauer & Sandholm, 2009]
- Contractive games [Sandholm, 2015]
- Dissipative games [Sorin & Wan, 2016]

Convergence to equilibrium

Different behavior under **strict** monotonicity:

$$\frac{1}{2}\|X_{n+1}-x^*\|^2 = \frac{1}{2}\|X_n - x^*\|^2 - \gamma_n \underbrace{\langle V(X_n), X_n - x^*\rangle}_{>0 \text{ if } X_n \text{ not Nash}} + \frac{1}{2}\underbrace{\gamma_n^2 \|V(X_n)\|^2}_{\text{discretization error}}$$

Can the drift overcome the discretization error?

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Can the drift overcome the discretization error?

Theorem (M & Zhou, 2019)

- **Assume:** *strict monotonicity; any of* (A); *any of* (B); (C)
- **Then:** *any generalized Robbins-Monro learning algorithm converges to the game's (unique) Nash equilibrium with probability* 1

> In strictly monotone games, gradient methods $\rightsquigarrow$ Nash equilibrium

Outline

Overview

From flows to algorithms

From algorithms to flows

Flows in games

Monotone games

Spurious limits

Almost bilinear games

Consider the "almost bilinear" game

$$\min_{x_1 \in \mathcal{X}_1} \max_{x_2 \in \mathcal{X}_2} \quad L(x_1, x_2) = x_1 x_2 + \varepsilon \phi(x_2)$$

where $\varepsilon > 0$ and $\phi(x) = (1/2)x^2 - (1/4)x^4$

Properties:

- Unique critical point at the origin
- Not Nash; unstable under (MD)
- (MD) attracted to unique, stable limit cycle from almost all initial conditions

[Hsieh, M & Cevher, 2020]

Spurious limits in almost bilinear games

Trajectories of (RM) converge to a spurious cycle that contains no critical points

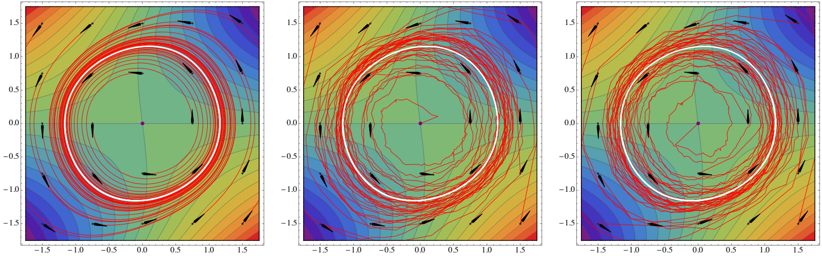

Figure: Left: (MD); center: SGD; right: stochastic extra-gradient (SEG)

Forsaken solutions

Another almost bilinear game

$$\min_{x_1 \in \mathcal{X}_1} \max_{x_2 \in \mathcal{X}_2} \quad L(x_1, x_2) = x_1 x_2 + \varepsilon[\phi(x_1) - \phi(x_2)]$$

where $\varepsilon > 0$ and $\phi(x) = (1/4)x^2 - (1/2)x^4 + (1/6)x^6$

Properties:

- Unique critical point at the origin
- Local Nash equilibrium; stable under (MD)
- Two isolated periodic orbits:
  - One unstable, shielding equilibrium, but small
  - One stable, attracts all trajectories of (MD) outside small basin

[Hsieh, M & Cevher, 2020]

Forsaken solutions in almost bilinear games

With high probability, (RM) forsakes the game's unique (local) equilibrium

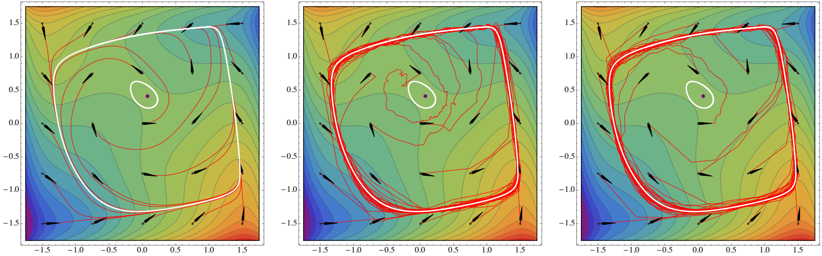

Figure: Left: (MD); center: SGD; right: SEG

The limits of gradient-based learning in games

Limit cycles $\implies$ internally chain transitive (ICT) = invariant, no proper attractors

Examples of ICT sets

- $V = \nabla f \implies$ components of critical points
- $L(x_1, x_2) = x_1 x_2 \implies$ any annular region centered on $(0, 0)$
- Almost bilinear $\implies$ isolated periodic orbits + unique stationary point

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Theorem (Hsieh, M & Cevher, 2020)

- **Assume:** *any of* **(A)**; *any of* **(B)**; **(C)**
- **Then:**
  - X_n *converges to an ICT of* (MD) *with probability* 1
  - (RM) *converges to attractors of* (MD) *with arbitrarily high probability*

Conclusions

In contrast to single-agent problems (optimization), game-theoretic learning

- May have limit points that are neither stable nor stationary
- Cannot avoid spurious, non-equilibrium points with positive probability
- Requires drastically different approach (mixed-strategy learning,…)

In memoriam

Bill Sandholm, 1970–2020