POLAR DECONVOLUTION OF MIXED SIGNALS

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1-WORLD OPTIMIZATION SEMINAR

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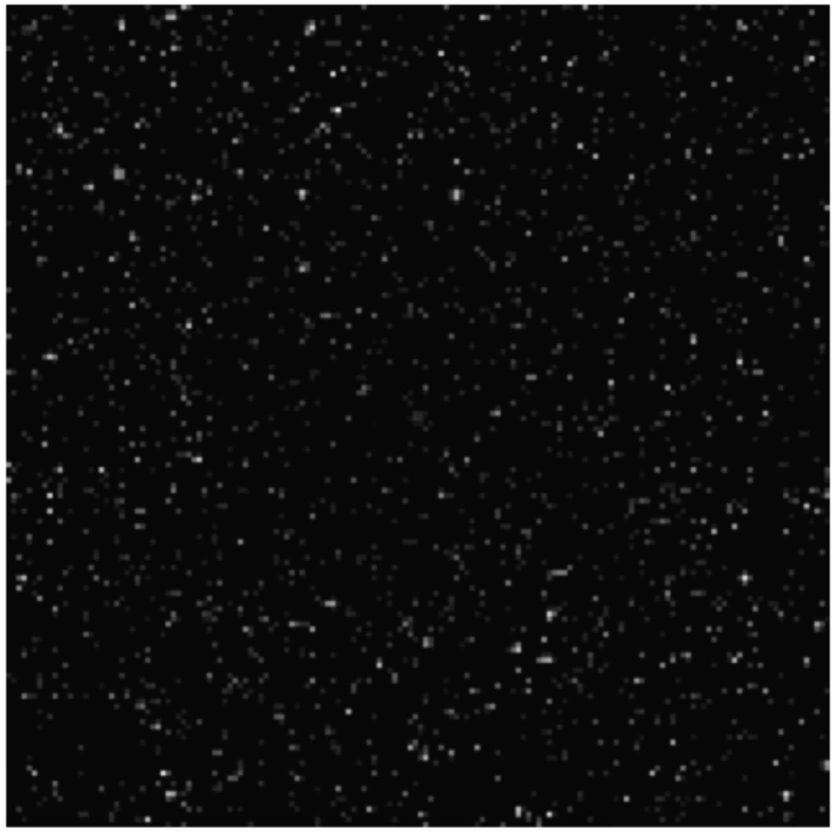
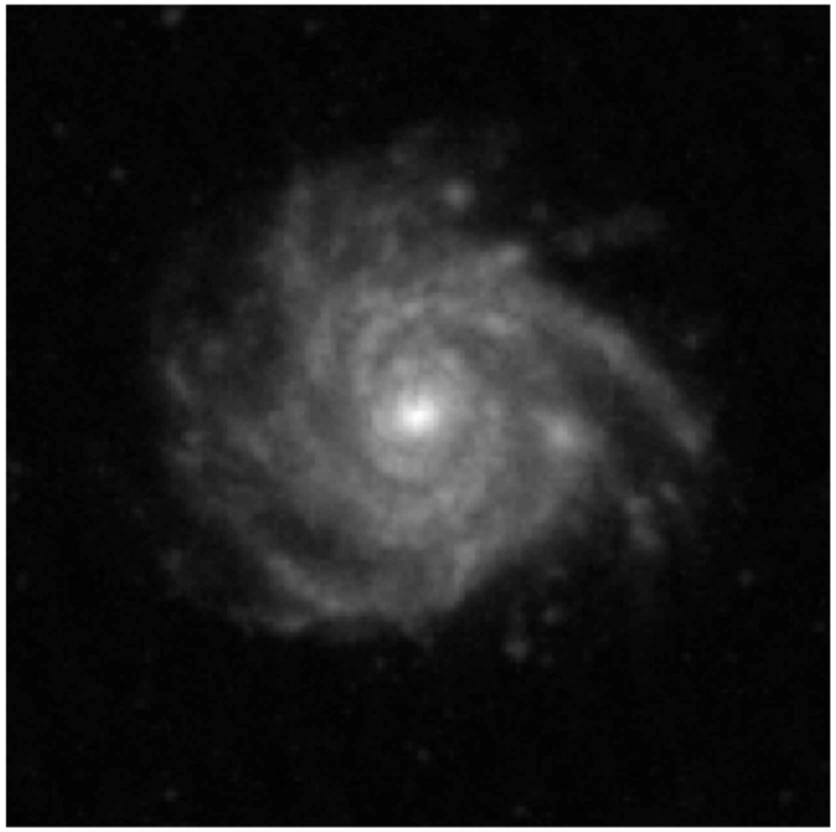
separating spikes and sinusoids

observation

sparse



sparse-in-frequency

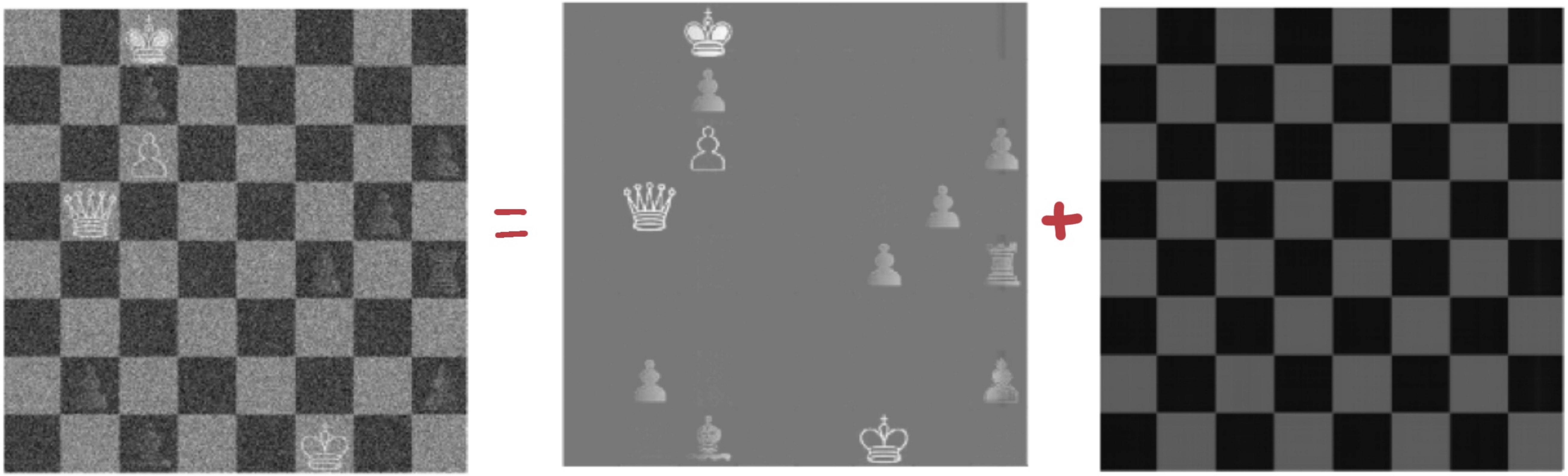


$$x = x_1 + x_2$$

$$= \sum_{i,j} e_i e_j^T + \sum_{i,j} D_2 e_i e_j^T$$

Chen, Donoho, Saunders (98), Donoho & Huo (01)

separating background - foreground - noise



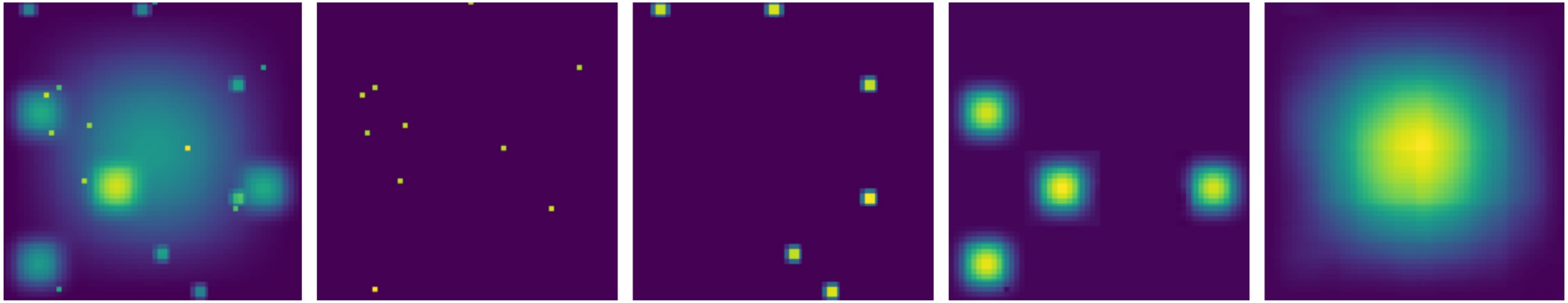
$$b_{\text{image}} + \text{noise} = X_{\text{sparse}} + X_{\text{Low-rank}}$$

$$= \sum_{a\in\mathcal{A}_1} c_a \cdot a + \sum_{a\in\mathcal{A}_2} c_a \cdot a$$

$$\mathcal{A}_1 = \{\pm e_i e_j^T\} \qquad \mathcal{A}_2 = \{uv^T \mid \|u\| = \|v\| = 1\}$$

Chandrasekaran et al (09) ; Candés et al. (09)

multiscale low-rank decomposition



$$X = X_1 + X_2 + X_3 + X_4$$

$$X_i = \sum_{a \in \mathcal{A}_i} c_a \cdot a$$

$$\mathcal{A}_i = \{ uv^T \mid u, v \in \mathbb{R}^{4^{i-1}}, \|u\| = \|v\| = 1 \}$$

Ong and Lustig (2016)

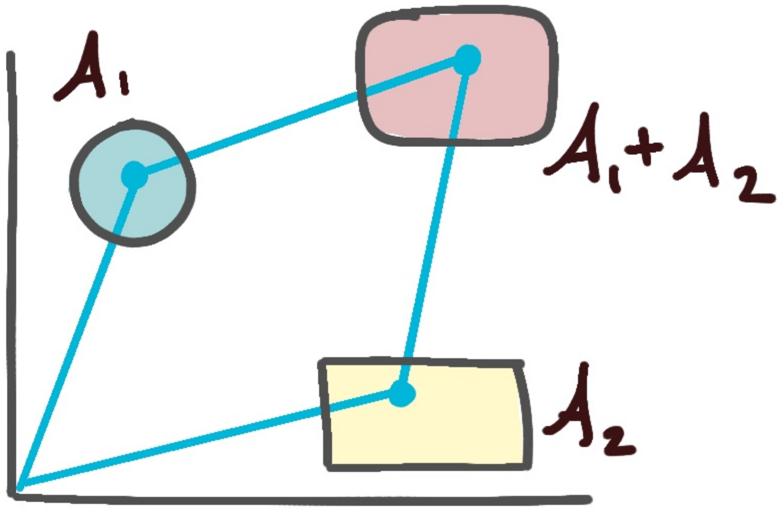
STORY LINE

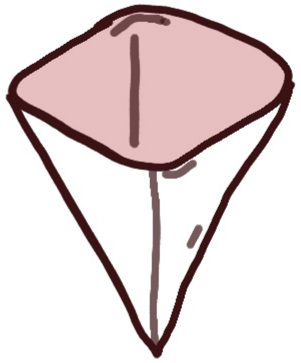
- signal demixing

$$b = M(x_1 + x_2 + \cdots + x_k) + \text{noise}$$

follow McCoy & Tropp (2013, 2014) analysis

- polar convolution





- conditional gradients

signal demixing model

$$b = M x_s^\natural + \text{noise} \qquad x_s^\natural = \sum_{i=1}^{K} x_i^\natural$$

sparse atomic decomposition for each component

$$x_i^\natural \in \mathbb{R}_+ \operatorname{conv} \mathcal{A}_i \qquad \mathcal{A}_i = \{\text{atomic set}\} \subseteq \mathbb{R}^n$$

examples

- sparse n-vector $\quad x = \sum_j^p c_j e_j \quad \mathcal{A} = \{\pm e_1, \dots, \pm e_n\}$
- low-rank symm matrices $\quad X = \sum_j^p \lambda_j u_j u_j^T \quad \mathcal{A} = \{ uu^T \mid \|u\|_2 = 1 \}$

morphological component analysis (Bobin et al.'07)

- constituent signals represent different morphologies
- "incoherence" between morphologies required

$x_s^\natural = x_1^\natural + x_2^\natural + \cdots$ where $x_i^\natural$ is A_i-sparse

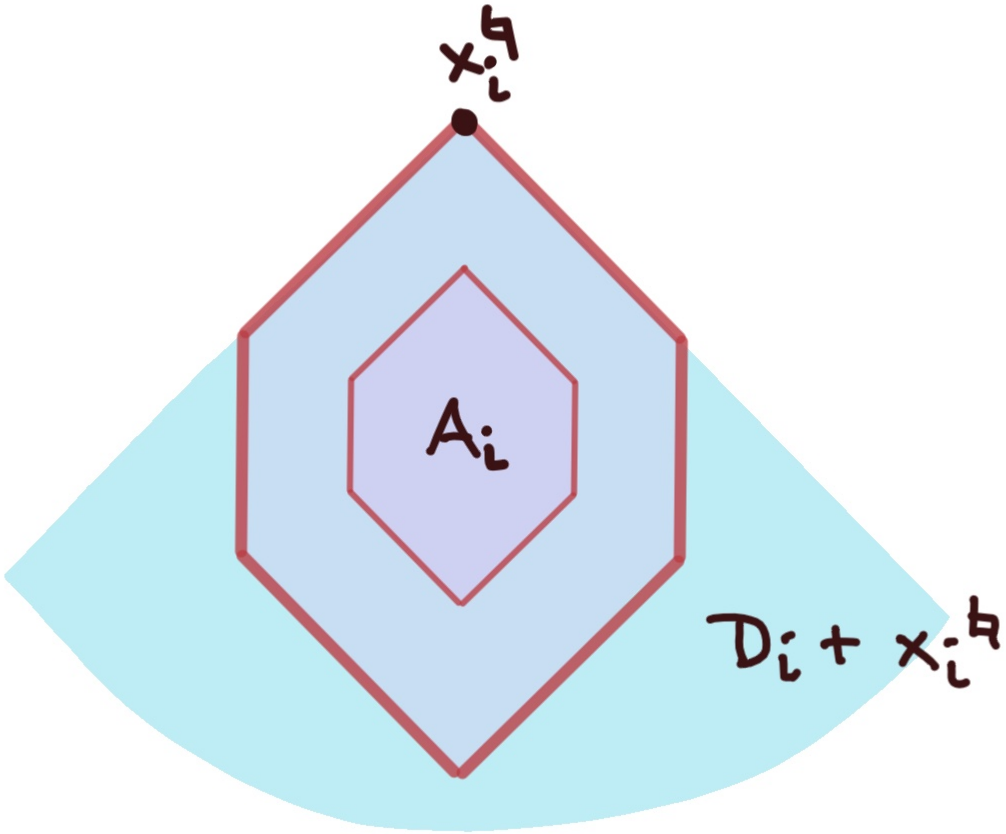
$\Downarrow$

signal components $x_i^\natural$ live on low-dim face of scaled atomic set

$\Downarrow$

"narrow" descent cone

$$D_i = \text{cone}\{d \mid \gamma_{A_i}(x_i^\natural + d) < \gamma_{A_i}(x_i^\natural)\}$$



gauge to atomic set A_i :

$$\gamma_{A_i}(x_i^\natural) = \inf\{\lambda \geq 0 \mid x \in \lambda \operatorname{conv} A_i\}$$

$$= \inf\{\textstyle\sum c_a \mid x = \sum c_a \cdot a,\ a \in A_i,\ c_a \geq 0\}$$

POLAR CONVOLUTION $x_s^\natural = x_1^\natural + x_2^\natural + \cdots$

$x_i^\natural$ is A_i-sparse $\Longrightarrow$ $x_s^\natural$ is $(\sum_i A_i)$-sparse

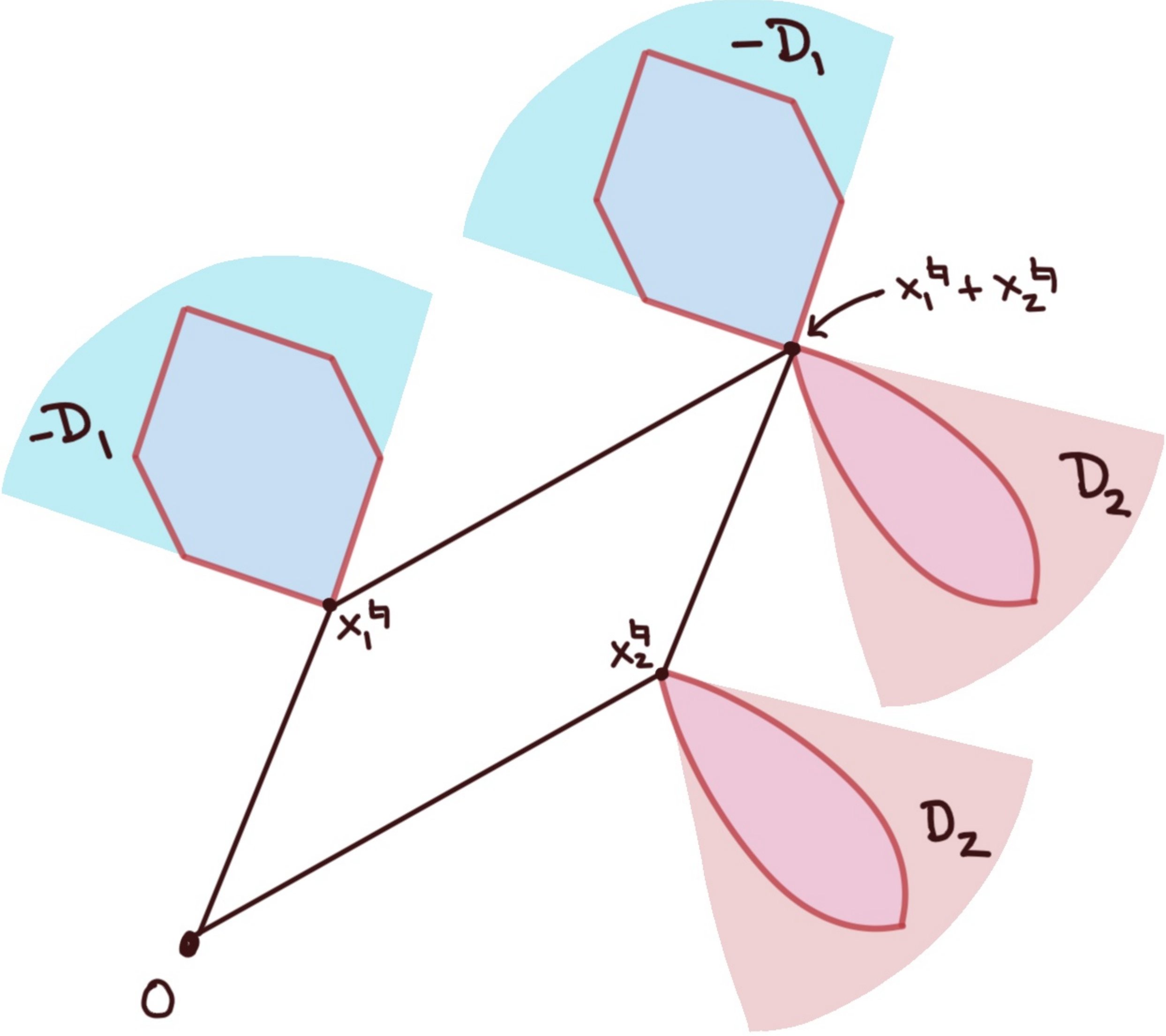
polar convolution [level-addit'n / Rockafellar '70 ; Seegar '11
F., Macêdo, Pong '19]

$$\gamma_{\sum_i A_i}(x_s^\natural) = \inf_{x_i} \left\{ \max_i \gamma_{A_i}(x_i) \;\middle|\; x_s^\natural = \sum_i x_i \right\} \qquad (P)$$

exact recovery :

$(x_1^\natural, x_2^\natural)$ unique sol'n
to (P) iff

$$-D_1 \cap D_2 = \{0\}$$



noisy demixing model

- $b = x_s^\natural + \text{noise}$ / $x_s^\natural = \sum_i x_i^\natural$ / $\|\text{noise}\|_2 \le \alpha$

polar deconvolution denoising

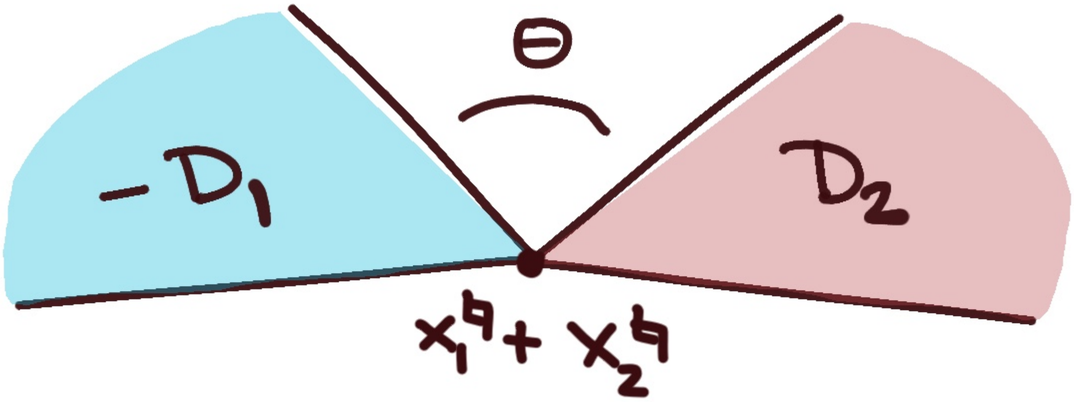
$$x_1^*, \dots, x_k^* \in \underset{x_i}{\operatorname{argmin}} \left\{ \max_i \gamma_{A_i}(x_i) \mid \left\| \sum_i x_i - b \right\|_2 \le \alpha \right\}$$

signal incoherence : $x_i^\natural$ are β-incoherent if $\forall i$,

$$\cos \angle \left(-D_i, \sum_{i \ne j} D_j\right) \le 1 - \beta \qquad \beta \in (0, 1]$$

proposition

$$\|x_i^* - x_i^\natural\|_2 \le \alpha / \sqrt{\beta}$$

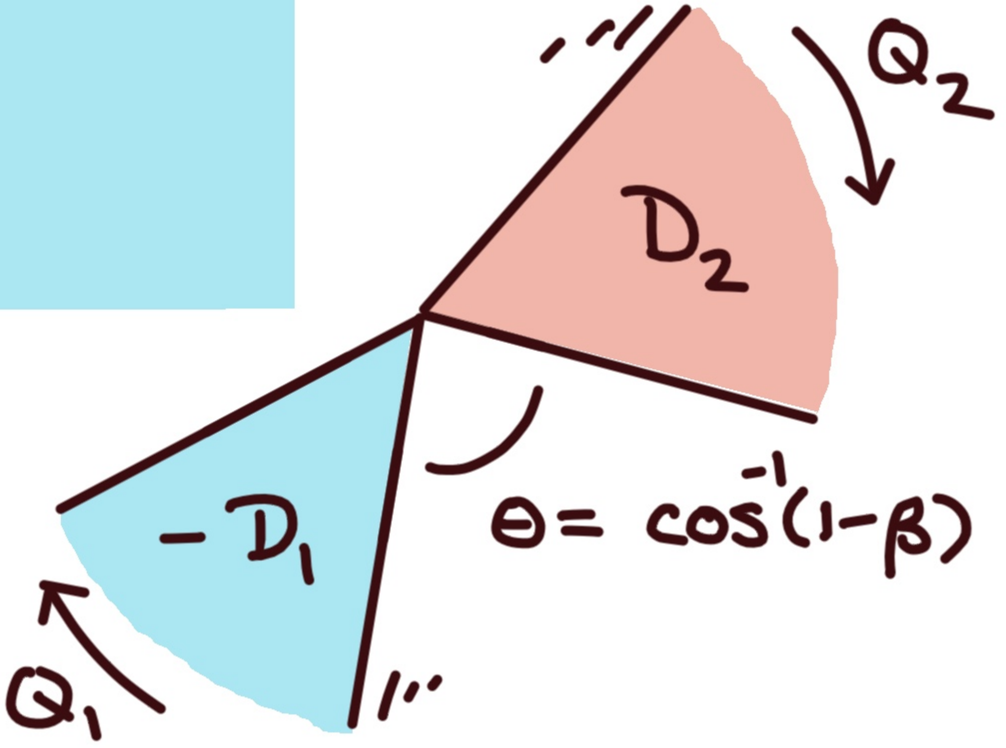


incoherence via random rotation [McCoy & Tropp '13]

$$b = x_s^\natural + \eta \quad / \quad x_s^\natural = \sum_{i}^{K} Q_i x_i^\natural \quad / \quad \|\eta\|_2 \le \alpha$$

Prop. under random rotations Q_i, descent cones $\{(Q_i x_i^\natural, Q_i A_i)\}_{i=1}^{K}$ are β-incoherent w.h.p., where

$$\beta := \frac{1}{2} - c\sqrt{\Delta/n}$$



total conic "width":

$$\Delta = \sum_{i}^{K} \text{stat.dim}(D_i)$$

$$\text{stat.dim}(D_i) = \mathbb{E}_g \|\text{proj}_{D_i}(g)\|^2 = \mathbb{E}_g \, \text{dist}^2_{D_i^\circ}(g)$$

$g \sim \text{normal}(0, I_n)$

random measurement model

$$b = M x_s^\natural + \eta \quad / \quad x_s^\natural = \sum_{i}^{K} x_i^\natural \quad / \quad M = \overset{n}{\boxed{}}\, m$$

equlibrated atomic sets :

$$\lambda_i := \gamma_{A_i}(x_i^\natural) / \gamma_{A_1}(x_1^\natural) \qquad\qquad A_s := \sum_{i}^{K} \lambda_i A_i$$

two-stage process

① decompression to recover superposition

$$x_s^* \in \operatorname{argmin} \{ \gamma_{A_s}(x) \mid \|Mx - b\|_2 \le \alpha \} \qquad \|\eta\| \le \alpha$$

② deconvolution to recover components

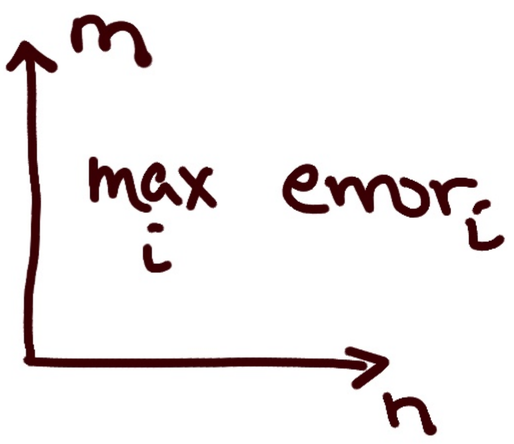
$$(x_1^* \cdots x_K^*) \in \operatorname*{argmin}_{x_i} \{ \max_i \gamma_{\lambda_i A_i}(x_i) \mid x_s^* = \sum x_i \}$$

recovery guarantees

theorem. if signals $\{x_i^\natural\}$ are randomly rotated, decompression / deconvolution produces whp

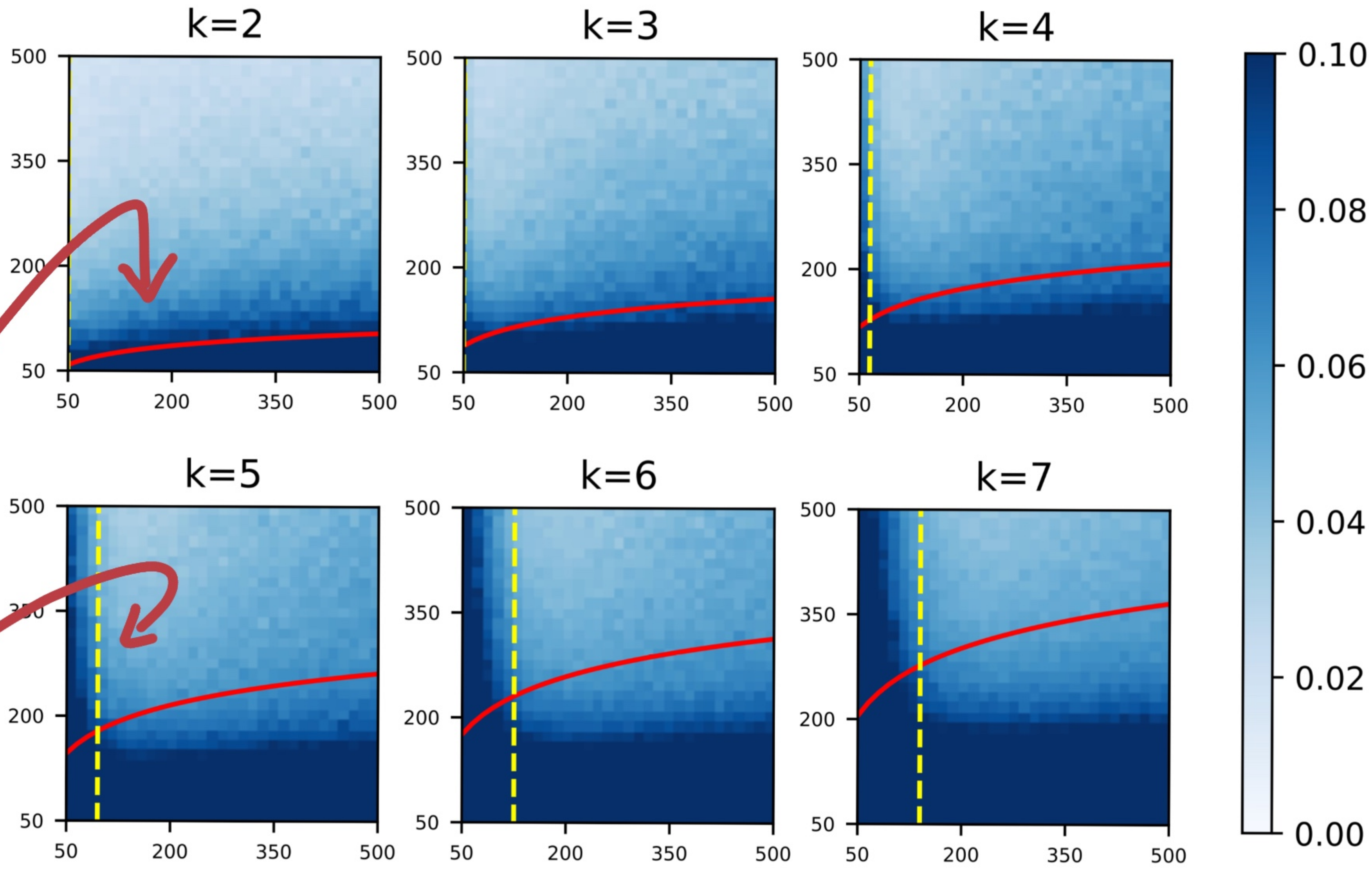
$$\|x_i^* - x_i^\natural\|_2 \le 2\alpha\left[\sqrt{\frac{1}{2} - c\sqrt{\frac{\Delta}{n}}}\cdot\left(\sqrt{m} - \sqrt{\Delta}\right)\right]_+^{-1}$$

errors for $x_1^* \dots x_k^*$ 5-sparse



min $\underline{m}$ req'd

min $\underline{n}$ req'd



CONDITIONAL GRAD

$$A_s = A_1 + \cdots + A_k$$

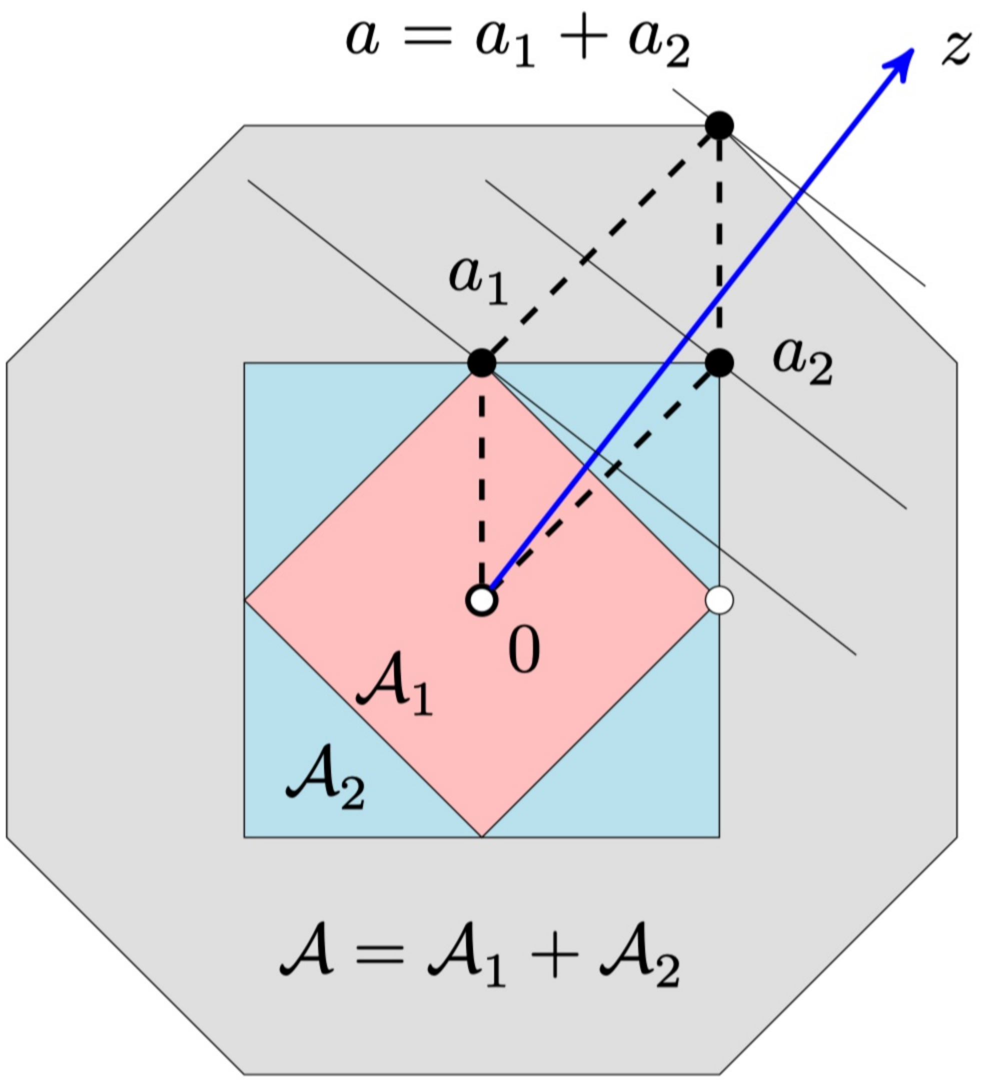
$$\min \{ f(x) \mid x \in \operatorname{conv} A_s \}$$

$$z^{(t)} := -\nabla f(x^{(t)})$$

$$a^{(t)} \in \operatorname{face}(A_s, z^{(t)})$$

$$A^{(t+1)} = A^{(t)} \cup \{a^{(t)}\}$$

$x^{(t+1)} \in \operatorname{conv} A^{(t+1)}$ ⇝ eg, linesearch



facial decomposition

$$\operatorname{face}\left(\sum_i A_i, z\right) = \sum_i \operatorname{face}(A_i, z)$$

ALGORITHM

$$\min \max_i \gamma_i(x_i)$$
$$\text{st} \quad M\left(\sum_i^K x_i\right) \approx b$$

iteration K :

$$z^{(t)} := M^* r^{(t)}$$

$$a_i^{(t)} :\in F_{A_i}(z^{(t)}) \qquad i=1,\cdots,K$$

$$\alpha_1, \alpha_2 := \underset{\alpha_i \in (0,1)}{\operatorname{argmin}} \left\| M \sum_i^K \alpha_i a_i^{(t)} - r^{(k)} \right\|_2$$

$$r^{(t+1)} := r^{(t)} + M \sum_i^K \alpha_i a_i^{(t)}$$

Then... (x_i, z^*) are A_i-aligned (easy recovery)

[Fan, Jeong, Sun, F. '20]

Complexity

O(data) space

$O(1/\varepsilon)$ iters for $\|M \sum_i^K x_i - b\| \leq \varepsilon$

FURTHER READING

(1) Friedlander, Macêdo, Pong "Gauge opt & duality"
SIOPT 2014

(2) Friedlander, Macêdo, "Gauge duality & low-rank spectral opt"
SISC 2016

(3) Aravkin, Burke, Drusvyatskiy, Friedlander, McPhee,
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SIOPT 2018

(4) Friedlander, Macêdo, Pong, "Polar Convolution"
SIOPT 2019

(5) Fan, Jeong, Sun, Friedlander, "Polar alignment &
atomic decomposition,
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(6) Fan, Jeong, Joshi, Friedlander, " Polar deconvolution of
mixed signals". arXiv 2010.10508, 2020