Resolvent Splitting with Minimal Lifting

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Joint work with Yura Malitsky (Linköping).

Reference for the talk:

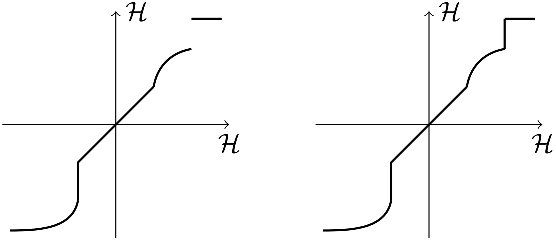
Resolvent splitting for sums of monotone operators with minimal lifting. Preprint: *arXiv:2108.02897*.

Monotone Operators

Let $\mathcal{H}$ be a real Hilbert space. An operator $B\colon \mathcal{H} \rightrightarrows \mathcal{H}$ is monotone if

$$\langle x - y, u - v \rangle \geq 0 \quad \forall (x, u), (y, v) \in \operatorname{graph} B.$$

A monotone operator B is maximally monotone if there exists no monotone operator whose graph properly contains $\operatorname{graph} B$.



Bauschke, H. H., & Combettes, P. L. (2011). *Convex analysis and monotone operator theory in Hilbert spaces*. New York: Springer.

Monotone Inclusions

Problem (*n*-operator monotone inclusion)

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \sum_{i=1}^{n} A_i(x),$$

where $A_i \colon \mathcal{H} \rightrightarrows \mathcal{H}$ is maximally monotone for all $i \in \{1, \ldots, n\}$.

Some important examples (potentially nonsmooth, finite sum):

- *Minimisation:* $A_i = \partial f_i$ for convex f_i gives

$$\min_x \sum_{i=1}^{n} f_i(x).$$

- *Minimax:* $A_i = \binom{\partial_u \Phi_i}{\partial_v(-\Phi_i)}$ for convex-concave $\Phi_i(u, v)$ gives

$$\min_u \max_v \sum_{i=1}^{n} \Phi_i(u, v).$$

(But examples are not the main focus of this talk.)

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Solving Monotone Inclusions ($n = 1$)

Recall that the resolvent of an operator $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is defined as

$$J_B := (\mathrm{Id} + B)^{-1}.$$

If B is maximally monotone, then J_B is a single-valued with full domain.

Proximal Point Algorithm

Let $A_1 : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone with $\operatorname{zer} A_1 \neq \emptyset$. Given $z^0 \in \mathcal{H}$, consider the sequence (z^k) given by

$$z^{k+1} = J_{A_1}(z^k) \quad \forall k \in \mathbb{N}.$$

Then $z^k \rightharpoonup z \in \operatorname{Fix} J_{A_1} = \operatorname{zer} A_1$.

Solving Monotone Inclusions ($n = 2$)

Douglas–Rachford Splitting

Let $A_1, A_2 \colon \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone with $\operatorname{zer}(A_1 + A_2) \neq \emptyset$. Given $z^0 \in \mathcal{H}$, consider the sequence (z^k) given by

$$z^{k+1} = T_{\mathrm{DR}}(z^k) := z^k + J_{A_2}\big(2J_{A_1}(z^k) - z^k\big) - J_{A_1}(z^k) \quad \forall k \in \mathbb{N}.$$

Then $z^k \rightharpoonup z \in \operatorname{Fix} T_{\mathrm{DR}}$ and $J_{A_1}(z^k) \rightharpoonup J_{A_1}(z) \in \operatorname{zer}(A_1 + A_2)$.

Note, two different sequences are involved:

- T_{DR} generates (z^k), but this sequence does not solve the problem.
- The resolvent J_{A_1} is applied to (z^k) to get the solution sequence.

Solving Monotone Inclusions ($n \geq 3$)

Let $A = (A_1, \ldots, A_n)$ be an n-tuple of maximally monotone operators.

Reformulate the n-operator inclusion as a two operator inclusion:

$$x \in \operatorname{zer}\left(\sum_{i=1}^{n} A_i\right) \subseteq \mathcal{H} \iff \mathbf{x} = (x, \ldots, x) \in \operatorname{zer}(N_{\Delta_n} + A) \subseteq \mathcal{H}^n,$$

where N_{Δ_n} is normal cone to the diagonal subspace given by

$$\Delta_n := \{(x_1, \ldots, x_n) \in \mathcal{H}^n : x_1 = \cdots = x_n\}.$$

Douglas–Rachford Splitting in the Product Space

Apply Douglas–Rachford splitting in $\mathcal{H}^n$ to the two operator inclusion involving N_{Δ_n} and A. The DR operator $T_{\mathrm{DR}} : \mathcal{H}^n \to \mathcal{H}^n$ can be expressed in terms of $J_A = (J_{A_1}, \ldots, J_{A_n})$ and

$$J_{N_{\Delta_n}}(\mathbf{z}) = P_{\Delta_n}(\mathbf{z}) = \left(\frac{1}{n}\sum_{i=1}^{n} z_i, \ldots, \frac{1}{n}\sum_{i=1}^{n} z_i\right).$$

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Goal of this Talk

- The literature is mostly devoted establishing what *is possible*:
  - Developing and analysing algorithms for solving monotone inclusions.
  - Algorithms distinguished based on the properties of $A_1, \dots, A_n$:
    - Set or single-valued, Lipschitz, cocoercive, strongly monotone, etc.

- Very little work concerned with examining what *is not possible*.
  - Statements like "There exists no algorithm with the following properties."
  - To be able to make such statements, we need to formalise the "rules".

⋆ **Main goal of talk belongs to the second category.**

Roughly speaking, our rules are:

- Fixed point algorithms which employ the resolvents of $A_1, \dots, A_n$.

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Structure of Resolvent Splitting Algorithms

Definitions in this section from:

Ryu, E. K. (2020). Uniqueness of DRS as the 2 operator resolvent-splitting and impossibility of 3 operator resolvent-splitting. *Mathematical Programming*, 182(1), 233-273.

Fixed Point Encodings

Let $\mathcal{A}_n$ denote the set of n-tuples of maximally monotone operators.
i.e., $A = (A_1, \ldots, A_n) \in \mathcal{A}_n$ when all A_i's are maximally monotone.

Definition (Fixed point encoding)

A pair of single-valued operators (T_A, S_A) is a fixed point encoding for $\mathcal{A}_n$ if, for all $A \in \mathcal{A}_n$, the following hold:

1. $\operatorname{Fix} T_A \neq \emptyset \iff \operatorname{zer}\left(\sum_{i=1}^n A_i\right) \neq \emptyset$.
2. $\mathbf{z} \in \operatorname{Fix} T_A \implies S_A(\mathbf{z}) \in \operatorname{zer}\left(\sum_{i=1}^n A_i\right)$.

In addition, a fixed point encoding is said to be convergent if:[1]

3. For all initial points $\mathbf{z}^0$, we have $\mathbf{z}^{k+1} = T_A(\mathbf{z}^k) \rightharpoonup \mathbf{z} \in \operatorname{Fix} T_A$.

To interpret this definition, it helps to keep the following in mind:

- The fixed point operator, T_A, is the basis for the iterative algorithm:

$$\mathbf{z}^{k+1} = T_A(\mathbf{z}^k) \quad \forall k \in \mathbb{N}.$$

- The solution operator, S_A, maps fixed points of T_A to solutions.

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Examples of Fixed Point Encodings

- Proximal point algorithm is a fixed point encoding for $\mathcal{A}_1$ with

$$T_A = J_{A_1} \quad \text{and} \quad S_A = \mathrm{Id}\,.$$

- Douglas–Rachford splitting is a fixed point encoding for $\mathcal{A}_2$ with

$$T_A = \mathrm{Id} + J_{A_2}(2J_{A_1} - \mathrm{Id}) - J_{A_1} \quad \text{and} \quad S_A = J_{A_1}\,.$$

- DR splitting in the product space is a fixed point encoding for $\mathcal{A}_n$ with

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where we note that, for this case, $T_A \colon \mathcal{H}^n \to \mathcal{H}^n$ and $S_A \colon \mathcal{H}^n \to \mathcal{H}$.

⋆ All these examples are actually convergent fixed point encodings.

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An Example which is not a Fixed Point Encoding

- The forward-backward algorithm given by

$$T_{\mathrm{FB}} = J_{\lambda A_1}\left(\mathrm{Id} - \lambda A_2\right)$$

  is **not** a convergent fixed point encoding for $\mathcal{A}_2$ because:
  - Convergence of fixed point iteration requires A_2 to be cocoercive.
  - T_{FB} is single-valued only when A_2 is single-valued.

- To be a fixed point encoding for $\mathcal{A}_2$, the properties must hold **for all** pairs of maximally monotone operators $A = (A_1, A_2) \in \mathcal{A}_2$.

Resolvent Splitting and Frugality

Definition (Resolvent splitting and frugality)

A fixed point encoding (T_A, S_A) is a resolvent splitting if, for all $A \in \mathcal{A}_n$, there is a procedure that evaluates T_A and S_A at a point that uses only:

1. Vector addition.
2. Scalar multiplication.
3. The resolvents $J_{A_1}, \ldots, J_{A_n}$.

In addition, if the procedure uses each resolvent only once, then the resolvent splitting is said to be frugal.

Implications for T_A and S_A:

- Allows for a kind of canonical form in terms of coefficient matrices.
- Informally, non-linearities in T_A can only arise from the resolvents.

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Examples for (Frugal) Resolvent Splittings

- All fixed point encoding on previous slide are frugal resolvent splittings.
- The convergent fixed point encoding (T_A, S_A) for $\mathcal{A}_1$ given by

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  is a resolvent splitting but it is **not** a frugal resolvent splitting.
- Methods whose iterations project onto separating hyperplanes are **not** resolvent splittings, even though they use $J_{A_1}, \dots, J_{A_n}$.
  - Haugazeau-type methods, Projective splitting (Eckstein–Svaiter '08), etc.
- Methods whose iterations use the resolvents $J_{\lambda_1 A_1}, \dots, J_{\lambda_n A_n}$ with different values for $\lambda_1, \dots, \lambda_n > 0$ are **not** resolvent splittings
  - Parallel Douglas–Rachford with reduced dimension (Condat–Kitaharra–Contreras–Hirabayashi '20, Campoy '21).

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The Structure of the Solution Map

Recall that a fixed point encoding must satisfy:

$$\mathbf{z} \in \operatorname{Fix} T_A \implies S_A(\mathbf{z}) \in \operatorname{zer}\left(\sum_{i=1}^{n} A_i\right).$$

Using the canonical form of a frugal resolvents splitting, properties about the general structure of all such algorithms can be derived.

Proposition (Malitsky–T.)

Let (T_A, S_A) be a frugal resolvent splitting for $\mathcal{A}_n$. Suppose $\mathbf{z} \in \operatorname{Fix} T_A$ and let y_i denote the point where J_{A_i} is evaluated in the procedure for evaluating $T_A(\mathbf{z})$. Then, necessarily, we have

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Lifting

Definition (Lifting)

Let $d \in \mathbb{N}$. A fixed point encoding (T_A, S_A) is a *d-fold lifting* for $\mathcal{A}_n$ if $T_A \colon \mathcal{H}^d \to \mathcal{H}^d$ and $S_A \colon \mathcal{H}^d \to \mathcal{H}$.

- Value of d represents number of copies of variable needed to use T_A.
- Smaller d means the corresponding algorithm needs less memory.
- Methods which attain the smallest value of d (for a given n) are said to have "minimal lifting". That is, they have the lowest memory requirements for algorithm class.

$\star$ **We focus on (minimal) lifting for frugal resolvent splittings.**

Examples of Lifting for Frugal Resolvent Splittings

- Proximal point algorithm for $\mathcal{A}_1$ has 1-fold lifting (*i.e.*, no lifting).
- Douglas–Rachford splitting for $\mathcal{A}_2$ has 1-fold lifting (*i.e.*, no lifting): $T_A\colon \mathcal{H} \to \mathcal{H}$ where

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Ryu's Splitting Algorithm for $n = 3$

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Theorem (Ryu, Aragón–Campoy–T.)

Let $A_1, A_2, A_3 \colon \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone with $\operatorname{zer}(\sum_{i=1}^3 A_i) \neq \emptyset$. Given an initial $\mathbf{z}^0 \in \mathcal{H}^2$, define the sequences $(\mathbf{z}^k)$ and $(\mathbf{x}^k)$ as above. Then the following assertions hold.

1. $\mathbf{z}^k \rightharpoonup \mathbf{z} \in \operatorname{Fix} T_A$.
2. $\mathbf{x}^k \rightharpoonup (x,x,x) \in \mathcal{H}^3$ with $x \in \operatorname{zer}(\sum_{i=1}^3 A_i)$.

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Conjecture

The minimal amount of lifting is given by $d^*(n) = n - 1$ for $n \geq 2$.

Resolvent Splitting with Lifting

As a first step towards resolving the conjecture, we showed the following.

Theorem (Malitsky-T.)

Let $n \geq 2$. If (T_A, S_A) is a frugal resolvent splitting for $\mathcal{A}_n$ with d-fold lifting, then $d \geq n-1$.

- Proof is by contradiction and uses the rank-nullity theorem applied to the coefficient matrices in the canonical form of T_A.
- No need to consider S_A directly – already determined by proposition.
- **Consequence of theorem:** $d^*(n) = n-1$ or $d^*(n) = n$.

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Frugal Resolvent Splitting with Minimal Lifting

Extending Ryu's Splitting from $n = 3$ to $n = 4$

We tried (unsuccessfully) to extend Ryu's scheme to $n \geq 4$ operators.

Let $\gamma \in (0, 1)$. Define $T : \mathcal{H}^2 \to \mathcal{H}^2$ according to

$$T(\mathbf{z}) = \mathbf{z} + \gamma \begin{pmatrix} x_4 - x_1 \\ \\ x_4 - x_3 \end{pmatrix} \quad \text{where} \quad \begin{cases} x_1 = J_{A_1}(z_1) \\ x_2 = J_{A_2}(z_2 + x_1) \\ \\ x_4 = J_{A_4}(x_1 - z_1 + x_2 - z_2) \end{cases},$$

- Possible four operator extension of Ryu's splitting (red terms new).
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Searching for New Methods

Let $\mathbf{z} = (z_1, \ldots, z_{n-1}) \in \mathcal{H}^{n-1}$ and $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{H}$.

Consider candidates for the operator T_A of the form

$$T_A(\mathbf{z}) = \mathbf{z} + \gamma M \mathbf{x} \quad \text{where} \quad x_i = J_{A_i}(y_i) \quad \forall i \in \{1, \ldots, n\}$$

for some matrix $M \in \mathbb{R}^{(n-1)\times n}$.

Assuming T_A is a frugal resolvent splitting, we can deduce that:

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So after choosing such an M, we need to investigate expressions for $\mathbf{y}$:

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$$\begin{cases} x_1 = J_{A_1}(z_1), \\ x_i = J_{A_i}(z_i + x_{i-1} - z_{i-1}) \qquad \forall i \in \{2, \dots, n-1\} \\ x_n = J_{A_n}(x_1 + x_{n-1} - z_{n-1}). \end{cases}$$

- If $n = 2$, then T_A is the same as T_{DR} for A_1 and A_2.
- When $n = 3$, it is different to Ryu's splitting method.

The Fixed Point Set of T_A

Lemma (Malitsky–T.)

Let $n \geq 2$, $A = (A_1, \ldots, A_n) \in \mathcal{A}_n$ and $\gamma > 0$. Then we have:

1. $\operatorname{Fix} T_A \neq \emptyset \iff \operatorname{zer}\left(\sum_{i=1}^n A_i\right) \neq \emptyset$.
2. If $\mathbf{z} = (z_1, \ldots, z_{n-1}) \in \operatorname{Fix} T_A$, then $x \in \operatorname{zer}\left(\sum_{i=1}^n A_i\right)$ where

$$x := J_{A_1}(z_1) = J_{A_i}(z_i + x - z_{i-1}) = J_{A_n}(2x - z_{n-1}) \tag{1}$$

for all $i \in \{2, \ldots, n-1\}$.

- Shows that T_A can be used to define a fixed point encoding for $\mathcal{A}_n$.
- Any resolvents in (1) can be used solution map. For instance:

$$S_A(\mathbf{z}) := J_{A_1}(z_1).$$

- How about (weak) convergence of the fixed point iteration?

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Nonexpansivity Properties of T_A

Lemma (Malitsky–T.)

Let $n \geq 2$, $A = (A_1, \dots, A_n) \in \mathcal{A}_n$ and $\gamma > 0$. Then we have

$$\|T_A(\mathbf{z}) - T_A(\bar{\mathbf{z}})\|^2 + \frac{1-\gamma}{\gamma}\|(\mathrm{Id} - T_A)(\mathbf{z}) - (\mathrm{Id} - T_A)(\bar{\mathbf{z}})\|^2 + \frac{1}{\gamma}\Big\|\sum_{i=1}^{n-1}(\mathrm{Id} - T_A)(\mathbf{z})_i - \sum_{i=1}^{n-1}(\mathrm{Id} - T_A)(\bar{\mathbf{z}})_i\Big\|^2 \leq \|\mathbf{z} - \bar{\mathbf{z}}\|^2,$$

where $\mathbf{z} = (z_1, \dots, z_n) \in \mathcal{H}^{n-1}$ and $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n) \in \mathcal{H}^{n-1}$.

- If $\gamma \in (0,1)$, then the operator T_A is γ-averaged nonexpansive.
- Counter-example in paper: in general, we cannot take $\gamma = 1$.
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Main Convergence Theorem

Fixed point algorithm:

$$\mathbf{z}^{k+1} = T_A(\mathbf{z}^k) = \mathbf{z}^k + \gamma \begin{pmatrix} x_2^k - x_1^k \\ x_3^k - x_2^k \\ \vdots \\ x_n^k - x_{n-1}^k \end{pmatrix} \text{ where } \begin{cases} x_1^k = J_{A_1}(z_1^k), \\ x_i^k = J_{A_i}(z_i^k + x_{i-1}^k - z_{i-1}^k \\ x_n^k = J_{A_n}(x_1^k + x_{n-1}^k - z_{n-1}^k). \end{cases}$$

Theorem (Malitsky–T.)

Let $n \geq 2$, $A = (A_1, \dots, A_n) \in \mathcal{A}_n$ with $\operatorname{zer}\left(\sum_{i=1}^n A_i\right) \neq \emptyset$, and $\gamma \in (0,1)$. Given $\mathbf{z}^0 \in \mathcal{H}^{n-1}$, let $(\mathbf{z}^k) \subseteq \mathcal{H}^{n-1}$ and $(\mathbf{x}^k) \subseteq \mathcal{H}^n$ be given as above. Then, the following assertions hold.

1. $\mathbf{z}^k \rightharpoonup \mathbf{z} \in \operatorname{Fix} T_A$.
2. $\mathbf{x}^k \rightharpoonup (x, \dots, x) \in \mathcal{H}^n$ with $x \in \operatorname{zer}\left(\sum_{i=1}^n A_i\right)$.

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Some further refinements:

- If $A_2, \ldots, A_n$ are uniformly monotone (but not necessarily A_1), then $(\mathbf{x}^k)$ converges strongly. This holds in the limiting case $\gamma = 1$
  - In contrast, Peaceman–Rachford splitting (= limiting case of DR) in the product space requires all operators to be uniformly monotone.
- If A_i's are normal cones to subspaces S_i, then $(\mathbf{x}^k)$ converges strongly and $x := P_{\cap_{i=1}^n S_i}\left(\frac{1}{n-1}\sum_{i=1}^n z_i^0\right)$. (Bauschke–Singh–Wang)

Bauschke, H. H., Singh, S., & Wang, X. (2021). The splitting algorithms by Ryu and by Malitsky–Tam applied to normal cones of linear subspaces converge strongly to the projection onto the intersection. *arXiv:2109.11072*.

Minimal Lifting for Frugal Resolvent Splitting

Conjecture

The minimal amount of lifting is given by $d^*(n) = n - 1$ for $n \geq 2$.

Combining everything in this talk so far, gives the following answer.

Corollary (Malitsky–T.)

Suppose $n \geq 2$. There exists a convergent frugal resolvent splitting for $\mathcal{A}_n$ with $(n-1)$-fold lifting. Moreover, this is the minimal amount of lifting possible with frugal resolvent splittings for $\mathcal{A}_n$.

Algorithmic consequences:

- In general, it is not possible to do too much better than the product space: n-fold lifting vs $(n-1)$-fold lifting.
- For small n, the difference is more significant. For large n, less so.

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- New n-operator resolvent splitting method that generalises DR.

Directions for future work:

- Finer properties of new splitting algorithm (*e.g.*, inconsistent prob).
- How does frugality affect the amount of lifting needed? Trade off?
- Characterise all frugal resolvent splittings for n-operators?

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1	1	Proximal Point algorithm
2	1	Douglas–Rachford algorithm
3	2	Ryu's algorithm + This Work + Others?
$n \geq 2$	$n-1$	This Work + Others?

⋆ Perhaps we should more often examine what might *not be possible*

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