Hidden Convexity in Nonconvex Quadratic Optimization

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Nonconvex Quadratic Optimization

Data: $[Q_i \in S_n,\ b_i \in \mathbb{R}^n,\ c_i \in \mathbb{R}],\quad i \in [0, m]$

$$\begin{aligned} \inf \quad & x^T Q_0 x + 2b_0^T x \\ x^T Q_i x + 2b_i^T x + c_i & \leq 0,\ i \in [1, p] \\ x^T Q_i x + 2b_i^T x + c_i & = 0,\ i \in [p+1, m],\ x \in \mathbb{R}^n \end{aligned}$$

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Arise in many and disparate contexts.......

- Ill-posed problems/Regularization/Least Squares
- Eigenvalue perturbations
- Optimization algorithms: Trust Region Methods
- Polynomial Optimization problems
- Models for fundamentals Combinatorial/Graph Optimization problems (Max cut, stability number, max clique, etc..)
- Robust Optimization
- Spin Glasses....and much more....!

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A BRIDGE between: Continuous and Discrete Optimization....
Thus, not surprisingly so **"HARD"** to analyze/solve.

Outline

- Nonconvex optimization are generally non-tractable (NP hard).
- However, some classes of nonconvex problems can be solved.
- **Hidden convex** problems are nonconvex problems which admit an equivalent convex reformulation.

Focus on **detecting "Hidden Convexity"** in Nonconvex QP

- Duality
- Lifting and Semidefinite Relaxation
- Exact solutions for some classes of QP
- Convexity of the Image of a Quadratic Map
- The S-Procedure

Duality: A Quick Review

Data [f,C]: $f : \mathbb{R}^n \to (-\infty, +\infty], \; C \subseteq \mathbb{R}^n$

Primal Problem

$$(P) \qquad v(P) = \inf\{f(x) : \; x \in C\} \equiv \inf_{x \in \mathbb{R}^n}\{f(x) + \delta_C(x)\}$$

Dual Problem **Uses the same data**

$$(D) \qquad v(D) = \sup_y\{-f^*(y) - \delta_C^*(-y) : \; y \in \operatorname{dom} f^* \cap \operatorname{dom} \delta_C^*\};$$

with $f^*(y) := \sup_x\{\langle x, y\rangle - f(x)\}$; $\delta_C :=$ indicator of C.

Properties of (P)-(D)

- Dual is **always convex** (sup-concave)
- **Weak duality holds:** $v(P) \geq v(D)$ for any feasible pair (P)-(D)

Duality:Key Questions

$$\boxed{v(P) = \inf\{f(x) : \ x \in C\}; \quad v(D) = \sup_y\{-f^*(y) - \delta_C^*(-y)\}}$$

- **Zero Duality Gap:** when $v(P) = v(D)$?
- **Strong Duality:** when inf / sup attained?
- **Structure/Relations of Primal-Dual Optimal Sets/Solutions**

Convex $[f, C]$ + some Regularity Cond. deliver the answers

Less Popular: The Bidual

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The dual (DD) of (D) is then in term of the **bi-conjugate**:

$$(DD) \qquad v(DD) = \inf_z\{f^{**}(z) + \delta_C^{**}(z)\}$$

The dual (D) being *always convex*, one has (modulo some Reg. Cond.)

$$v(P) \geq v(D) = v(DD)$$

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$$v(P) \geq v(D) = v(DD)$$

- $v(DD)$ **is another lower bound for** $v(P)$
- $v(DD)$ natural **convexification** of $v(P)$ $\Longleftarrow$ $f^{**} \leq f$
- $v(DD)$ **often reveals hidden convexity – (or lack of)** in (P).

A Prototype in CO: Trust Region is Hidden Convex

The nonconvex trust region suproblem $[Q \in S_n, g \in \mathbb{R}^n, r > 0]$**:**

$$(TR) \quad \textbf{minimize} \ \{z^T Qz - 2g^T z : \ \|z\| \le r \ z \in \mathbb{R}^n\}$$

$Q \in S_n$ can be diagonalized, i.e., $\exists$ an orthogonal C

$$C^t QC = D := \text{diag}(d_1, \ldots, d_n), \ d_j \in \mathbb{R}, \ j = 1, \ldots, n; \ c := Cg$$

Theorem (Ben-Tal and T. (1996)). The nonconvex (TR) is equivalent to the convex problem

$$(CTR) \quad \min \left\{ \sum_{j=1}^{n} d_j y_j - 2|c_j|\sqrt{y_j} : \ \sum_{j=1}^{n} y_j \le r, \ y \in \mathbb{R}^n_+ \right\}.$$

More precisely, $\exists y^*$ **of** (CTR)**, and corresponding optimal solution of** (TR) **given by** $z^* = Cx^*, \ \ x_j^* = \textbf{sgn} c_j \sqrt{y_j^*}, \forall j$ **and** $\inf(TR) = \min(CTR)$
.

Proof. See more general results (e.g., min. of indefinite quadratic subject to 2-sided indefinite constraints; min of concave quadratic over fnitely many convex quadratic) proven via **biduality** in (Ben Tal-Teboulle (96)) □

A Prototype in DO: The Max-Cut Problem

Data Input: A graph $G = (V, E), V = \{1, 2, \ldots, n\}$ with weights $w_{ij} = w_{ji} \geq 0$ on the edges $(i, j) \in E$ and with $w_{ij} = 0$ if $(i, j) \notin E$.

Problem: Find the set of vertices $S \subset V$ that maximizes the weight of the edges with one end point in S and the other in its complement $\bar{S}$, i.e., to maximize the total weight across the cut $(S, \bar{S})$.

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Max Cut as a Nonconvex QP

$$(MC) \quad \max\{\sum_{i<j} w_{ij} \frac{1 - x_i x_j}{2} : \; x_i^2 = 1, \; i = 1, \ldots, n\}$$

Can be reformulated equivalently as

$$(MC) \qquad \max\{x^T Q x : \; x_i^2 = 1, \; i = 1, \ldots, n\}$$

where $S_n \ni Q := \frac{L}{4} \succeq 0, \;\; q_{ij} \leq 0 \, \forall i \neq j; \; L \equiv \text{diag}\, We - W;$
$e = (1, \ldots, n)^T$

Dual Representations of MC–(Shor–87)

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DUALITY IS "VERY FLEXIBLE"...

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The following are **"equal" upper bounds** for (MC)

- $\min_{u \in \mathbb{R}^n} \{u^T e : \; \mathrm{Diag}(u) \succeq Q\}$
- $\min_{u \in \mathbb{R}^n} \{u^T e + n\lambda_{\max}(Q - \mathrm{Diag}(u))\}$
- $\min_{u \in \mathbb{R}^n} \{n\lambda_{\max}(Q + \mathrm{Diag}(u)) : \; u^T e = 0\}$
- $\min_{u \in \mathbb{R}^n} \{u^T e : \lambda_{\min}(\mathrm{Diag}(u) - Q) \geq 0\}$

Notation:

For $u \in \mathbb{R}^n : \; \mathrm{Diag}(u) := \mathrm{Diag}(u_1, \ldots, u_n),$ Diagonal Matrix.

For $S^n \ni Z, \; \mathrm{diag}(Z) = (Z_{11}, \ldots, Z_{nn})^T \in \mathbb{R}^n$.

One More Dual Bound...The Bidual

$$(MC) \qquad \max\{x^T Qx : \; x_i^2 = 1, \; i = 1, \ldots, n\}$$

Using the (first) previous dual representation:

$$(DMC) \qquad \min_{u \in \mathbb{R}^n}\{u^T e : \; \mathrm{Diag}(u) \succeq Q\}$$

Take the dual of the above dual – The bidual:

$$(R) \qquad \max_{Z \in S^n}\{\mathrm{tr}\, QZ : \; \mathrm{diag}(Z) = e, \; Z \succeq 0\}$$

Here one has: $v(MC) \leq v(DMC) = v(R)$

The pair of convex problems (DMC)-(R) are "Semidefinite Optimization problems"

Semi-Definite Programming–SDP Relaxation

$$\min_{x \in \mathbb{R}^m} \{c^T x : A(x) \succeq 0\}; \quad \max_{Z \in S^n} \{\operatorname{tr} A_0 Z : \ \operatorname{tr} A_i Z = c_i, \ i \in [1, m] \ Z \succeq 0\}$$

where $A(x) := A_0 + \sum_{i=1}^m x_i A_i$

- SDP are special classes of **convex optimization** problems
- Computationally tractable: Can be **approximately solved** to a desired accuracy in polynomial time
- Naturally occurs in Relaxation of QP via **Lifting**

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DROP the Hard RANK ONE constraint=SDP RELAXATION

(R) $\equiv$ BIDUAL OF MC $\equiv$ SDP RELAXATION

Fundamental Question

A Fundamental Question in Nonconvex QP:

Tightness of the SDP relaxation ($\equiv$ Bidual Bound for QP)?

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A General Class of Nonconvex QP

$$(QP) \qquad v(QP) := \max\{x^T Q x : \ x^2 \in \mathcal{F}\}$$

$$x^2 \equiv (x_1^2, \ldots, x_n^2)^T; \ Q \in S_n; \ \ \mathcal{F} \subseteq \mathbb{R}^n, \text{ closed convex}$$

Extends the special case: [Max-Cut when $\mathcal{F} = \{e\}$].

A Trigonometric Representation of QP

Notation: $\forall X \in S_n$, $\arcsin X := \arcsin(X_{ij})$; $\langle A, B\rangle = \mathrm{tr}(AB)$
$\mathrm{diag}(X) := (X_{11}, \ldots, X_{nn})^T \in \mathbb{R}^n$, $D := \mathrm{Diag}(d_1, \ldots, d_n)$ diag. matrix.

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$$(TQP) \quad \max_{(d,X)}\{\frac{2}{\pi}\langle Q, D\arcsin(X)D\rangle : \; d \in \mathbb{R}^n_+, \; d^2 \in \mathcal{F}, \; X \succeq 0, \mathrm{diag}(X) = e\}$$

Theorem (Goemeans-Williamson 95, Nesterov 97, Ye 99)

$$\mathbf{v}(\mathbf{QP}) = \mathbf{v}(\mathbf{TQP})$$

(TQP) is the key tool to derive $(0, 1] \ni \rho$-*approximate* solutions to (QP)
[$\rho = .878...$ **for (MC);** $\rho = 2^{-1}\pi$ **for** $Q \succeq 0$]

$$\rho v(R) \leq \mathbf{v}(\mathbf{TQP}) = \mathbf{v}(\mathbf{QP}) \leq v(R)$$

where (R) is a semidefinite relaxation of (QP) given by

$$(R) \qquad \max\{\langle Q, Z\rangle : \; \mathrm{diag}(Z) \in \mathcal{F}, S_n \ni Z \succeq 0\}.$$

Nonconvex QP with Exact Solutions

QP with Exact Solutions through their Convex relaxation counterpart?

- (QP) $\quad v(QP) := \max\{x^T Q x : \; x^2 \in \mathcal{F}\}$
- (R) $\quad \max\{\langle Q, Z\rangle : \; \operatorname{diag}(Z) \in \mathcal{F}, S^n \ni Z \succeq 0\}$

Are there (QP) for which v(R)=V(QP)?

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Theorem (Zhang 2000) Let $S^n \ni Q$ with $q_{ij} \geq 0 \, \forall i \neq j$. Then,

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If Z solves (R) then $\sqrt{\operatorname{diag} Z}$ solves (QP).

Proof based on the key (TQP) representation + some other approximation and penalty arguments.

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Proof based on the key (TQP) representation + some other approximation and penalty arguments.

We will show that **this can be proved directly via duality**.

A Bidual Approach to Exact Solutions

$$(QP) \qquad \max\{x^T Qx : \ x^2 \in \mathcal{F}\} \iff \max_{x,y}\{x^T Qx : \ y = x^2, y \in \mathcal{F}\}$$

- **A dual of (QP) is (DQP)**

$$(DQP) \quad \min_{u\in\mathbb{R}^n} \left\{ \max_{x\in\mathbb{R}^n} x^T(Q-U)x + \max_{y\in\mathcal{F}} \langle y, u\rangle \right\} = \min_{u\in\mathbb{R}^n}\{\sigma_{\mathcal{F}}(u) : \ Q-U \preceq 0\}$$

where $U := \text{Diag}(u_1, \ldots, u_n)$, $\sigma_{\mathcal{F}}(u) =: \max\{\langle u, y\rangle : \ y \in \mathcal{F}\} \equiv \delta^*_{\mathcal{F}}(u)$.

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- **Bidual: Dual of the dual DQP**

$$(D^2 QP) \quad \max_{Z \succeq 0}\{\langle Q, Z\rangle + \min_{u \in \mathbb{R}^n}\{\delta^*_{\mathcal{F}}(u) - \langle Z, U\rangle\} = \max_{Z \succeq 0}\{\langle Q, Z\rangle : \ \mathrm{diag}(Z) \in \mathcal{F}\}$$

and $(D^2 QP)$ is nothing else but (R).

- **Regularity Cond. holds for (DQP)** $\Longrightarrow v(DQP) = v(D^2 QP)$

A Simple Duality Proof (Pinar-T. (06))

- **Weak duality:** $v(QP) \leq v(DQP)$ for any feasible P-D.
- **By Regularity:** $v(DQP) = v(D^2QP)$, thus $v(QP) \leq v(D^2QP)$.
- **Ask for equality, i.e. when** $v(QP) \geq v(D^2QP)$?

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Thus, with x feasible for (QP) and Z optimal for (D^2QP) we have:

$$v(QP) \geq \sum_{i,j} q_{ij}x_ix_j = \sum_i q_{ii}Z_{ii} + 2\sum_{i<j} q_{ij}\sqrt{Z_{ii}Z_{ij}}, \text{ while}$$

$$v(D^2QP) = \sum_{i,j} q_{ij}Z_{ij} = \sum_i q_{ii}Z_{ii} + 2\sum_{i<j} q_{ij}Z_{ij},$$

$$\implies v(QP) - v(D^2QP) \geq 0 \iff \sum_{i \neq j} q_{ij}(\sqrt{Z_{ii}Z_{jj}} - Z_{ij}) \geq 0$$

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$$\implies v(QP) - v(D^2QP) \geq 0 \iff \sum_{i \neq j} q_{ij} (\sqrt{Z_{ii} Z_{jj}} - Z_{ij}) \geq 0$$

But $Z \succeq 0 \implies Z_{ii} \geq 0,\ Z_{ii} Z_{jj} \geq Z_{ij}^2\ \forall i \neq j$

Therefore since we assumed $q_{ij} \geq 0,\ \forall i \neq j$ we are done!

Quadratic Maps with Convex Images

An old and classical subject in Mathematics....
In the *complex space* goes back to:
Hausdorff-Toeplitz Theorem [1918]: The numerical range of a linear operator is closed and convex.

Explicitly, in finite dimension, with A, B, $n \times n$, **Hermitian matrices one has:** $\{(z^*Az, z^*Bz) : \ \|z\| = 1, z \in \mathbb{C}^n\} \subseteq \mathbb{R}^2$, **is closed convex**.

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The Real Case – Let $A, B \in S^n$**.**

Dines Theorem (1940) $\{(x^TAx, x^TBx) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$ is convex.

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Dines Theorem (1940) $\{(x^T Ax, x^T Bx) : \ x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$ is convex.

A Key Result: Brickman's Theorem (1961). If $n \geq 3$ then

$$B := \{(x^T Ax, x^T Bx) : \quad \|x\| = 1, \ x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2 \text{ is closed convex}$$

A very good survey: Uhlig (1979).
Note: Brickman's Theorem fails for $n = 2$

Convexity of Image of Quadratic Map: Extensions?

- Let $A_i \in S^n$ (indefinite)
- For $m \geq 3$, $q_i(x) = x^T A_i x$, $i \in [1, m]$; **Quadratic forms**
- For $q_i(x) = x^T A_i x - 2b_i^T x + c_i$; **Quadratic functions**
- $q : \mathbb{R}^n \to \mathbb{R}^m$, $q(x) \equiv (q_1(x), \ldots, q_m(x))^T$

Question: When is $q(\mathbb{R}^n)$ convex for quadratic forms/functions?

Why do we care about the convexity of $q(\mathbb{R}^n)$?

At the root of fundamental questions/answers for *"Quadratic Problems"*

A. Quadratic Optimization e.g., Duality, Optimality, SDP...
B. Matrix related questions e.g., Simultaneous diag.
C. The S-Procedure
D. Computational Tractability

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Example: $(Q) \quad \min\{q_0(x) : \; q_i(x) \le 0, \; i = 1, \ldots, m, x \in \mathbb{R}^n\}$.

Let $W := \{(q_0(x), \ldots, q_m(x)) : \; x \in \mathbb{R}^n\}$.

Then (Q) equivalent to:

$$\min\{s_0 : s_i \le 0, \; i = 1, \ldots, m, s \in W\}$$

(Q) is convex iff the image W convex.

Quadratic Maps with Convex Images - Polyak (98)

- **Homogeneous Quadratic Forms: m=3**

$$\text{Let } S^n \ni A_i,\ q_i(x) = x^T A_i x,\ i = 1, 2, 3.$$

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Theorem 1 Suppose $n \geq 3$ and there exists $\mu \in \mathbb{R}^3$ such that

$$\mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3 \succ 0.$$

Then $\{(q_1(x), q_2(x), q_3(x)) :\ x \in \mathbb{R}^n\} \subseteq \mathbb{R}^3$ is closed convex.

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- **Nonhomogeneous Case : m=2** Let q_i be quadratic functions, i.e.,

$$q_i(x) = x^T A_i x - 2b_i^T x + c_i, \quad b_i \in \mathbb{R}^n, c_i \in \mathbb{R}, A_i \in S^n,\ i = 1, 2.$$

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Theorem 2 Suppose $n \geq 2$ and there exists $\mu \in \mathbb{R}^2$ such that

$$\mu_1 A_1 + \mu_2 A_2 \succ 0.$$

Then $\{(q_1(x), q_2(x)) : \ x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$ is closed convex.

The proofs rely on the nontrivial Brickman's theorem.
We will give a simple direct proof of Theorem 2.

Problem Reformulation

Given $q_i(x) = x^T A_i x - 2b_i^T x + c_i, \quad i = 1,2, \quad q(x) = (q_1(x), q_2(x))^T$
We want to prove that for all $n \geq 2$:

$$(*)\ \exists \mu \in \mathbb{R}^2 \text{ s.t. } \mu_1 A_1 + \mu_2 A_2 \succ 0 \implies q(\mathbb{R}^n) \text{ convex}$$

Under (*) we can assume $A_2 \succ 0$. Thus,

$$q(\mathbb{R}^n) := \{(q_1(x), ||x||^2)^T : x \in \mathbb{R}^n\} = \{(s,t) : s = q_1(x), t = ||x||^2\}.$$

For any $n \geq 2$, the sphere $S_t := \{x : \ ||x||^2 = t\}$ is connected. Thus,[1]

$$q(\mathbb{R}^n) = \{(s,t) : \inf_{x \in S_t} q_1(x) \leq s \leq \sup_{x \in S_t} q_1(x),\ t \geq 0\}$$

Therefore $q(\mathbb{R}^n)$ will be convex if we can show that

$$l(t) := \inf_{x \in S_t} q_1(x) \ \textbf{convex in } \ t \ (u(t) := \sup_{x \in S_t} q_1(x) \ \textbf{concave in } t)$$

[1] Let C be a connected subset of $\mathbb{R}^n$. Then any real valued function f defined and continuous on $\mathbb{R}^n$ attains in C every value between $\inf_{x \in C} f(x)$ and $\sup_{x \in C} f(x)$.

Direct Proof of Theorem 2

Lemma For $t \geq 0$ define the function

$$\ell(t) := \min\{q_1(x) : \ x \in S_t\} \equiv \min\{x^T A_1 x - 2b_1^T x : \ x \in S_t\}.$$

Then, $\ell(\cdot)$ **is a convex function on** $\mathbb{R}_+$**.**

Proof. via biduality ! □

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Proof. via biduality ! □

- Applying the Lemma implies that for all $n \geq 2$,

$$q(\mathbb{R}^n) = \{(s,t) : \inf_{x \in S_t} q_1(x) \leq s \leq \sup_{x \in S_t} q_1(x), \; t \geq 0\}$$

$$q(\mathbb{R}^n) = \{(s,t) : \; \ell(t) \leq s \leq u(t), t \geq 0\} \text{ **is convex in** } \mathbb{R}^2.$$

- **Note: Importance of dimension.** For $n = 1$, Lemma remains true.. but the set $(S_t = \{x \in \mathbb{R} : \; |x| = 1\})$ **is not connected!**

A General Nonconvex Class: Ratio of QP

Minimizing ratio of indefinite quadratic functions over an Ellipsoid

$$(RQ) \quad f_* := \inf \left\{ \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} : \|\mathbf{L}\mathbf{x}\|^2 \leq \rho \right\}$$

$$f_i(\mathbf{x}) = \mathbf{x}^T \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^T \mathbf{x} + c_i, i = 1, 2$$

$$\mathbf{A}_i = \mathbf{A}_i^T \in \mathbb{R}^{n \times n}, \mathbf{b}_i \in \mathbb{R}^n, c_i \in \mathbb{R}, \mathbf{L} \in \mathbb{R}^{r \times n}, \rho > 0$$

The feasible set

$$\mathcal{F} := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{L}\mathbf{x}\|^2 \leq \rho\}$$

represents a (possibly degenerate) ellipsoid .

Assumption: Problem (RQ) is well defined, i.e., $f_2(\mathbf{x}) > 0$ for every $\mathbf{x} \in \mathcal{F}$.

Motivation Arises in Estimation Problems: Regularized Total LS $Ax \approx b$ with (A, b) noisy data.

• Regularized Total Least Squares Problem (RTLS):
$f_1(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2, f_2(\mathbf{x}) = \|\mathbf{x}\|^2 + 1$ (both nice convex functions...but ratio is not!)

Attainment of the Minimum for RQ

Theorem The minimum of problem (RQ) is attained if either the feasible set is compact – or when $r < n$ if the following holds:

$$[SC] \quad \lambda_{\min}(\mathbf{M}_1, \mathbf{M}_2) < \lambda_{\min}(\mathbf{F}^T\mathbf{A}_1\mathbf{F}, \mathbf{F}^T\mathbf{A}_2\mathbf{F}),$$

where

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{F}^T\mathbf{A}_1\mathbf{F} & \mathbf{F}^T\mathbf{b}_1 \\ \mathbf{b}_1^T\mathbf{F} & c_1 \end{pmatrix}, \mathbf{M}_2 = \begin{pmatrix} \mathbf{F}^T\mathbf{A}_2\mathbf{F} & \mathbf{F}^T\mathbf{b}_2 \\ \mathbf{b}_2^T\mathbf{F} & c_2 \end{pmatrix}$$

$\mathbf{F}$ is an $n \times (n-r)$ matrix whose columns form an orthonormal basis for the null space of $\mathbf{L}$, and $\lambda_{\min}(\mathbf{A}, \mathbf{B}) := \max\{\lambda : \mathbf{A} - \lambda\mathbf{B} \succeq \mathbf{0}\}$

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[SC] plays a key role for establishing results in two directions:

- ✓ **An Exact SDP Relaxation for (RQ).**
- Convergence/complexity analysis of a fast algorithm for solving (RQ).
- Details in: A. Beck and M. Teboulle (2009).

An Exact SDP Relaxation of (RQ)– (Beck-T. (09))

Theorem Let $n \geq 2$ and suppose [SC] holds. Then,

$$\text{val (RQ)} = \text{val}(D), \quad \text{where (D) is given by}$$

$$\max_{\beta \geq 0, \alpha \in \mathbb{R}} \left\{ \alpha : \begin{pmatrix} \mathbf{A}_1 & \mathbf{b}_1 \\ \mathbf{b}_1^T & c_1 \end{pmatrix} \succeq \alpha \begin{pmatrix} \mathbf{A}_2 & \mathbf{b}_2 \\ \mathbf{b}_2^T & c_2 \end{pmatrix} - \beta \begin{pmatrix} \mathbf{L}^T\mathbf{L} & \mathbf{0} \\ \mathbf{0} & -\rho \end{pmatrix} \right\}$$

Proof relies on strong duality for homogeneous QP with two constraints + the attainability Condition [SC].

- The solution of (RQ) can be extracted from the solution of the semidefinite formulation.
- (RQ) belongs to the **privileged class of Hidden Convex Problems...**
- More results and details in Beck and Teboulle (2009, 2010).

The S-Procedure –(Yakubovitch-61)

$$q_i(x) = x^T Q_i x + 2b_i^T x + c_i, \quad Q_i \in S^n, \; i = 0, \ldots, m.$$

$$\mathcal{F} := \{x \in \mathbb{R}^n : \; q_i(x) \geq 0, \; i = 1, \ldots, m\}$$

Consider the following statements:

(S_1) $\quad q_0(x) \geq 0 \quad \forall x \in \mathcal{F}$

(S_2) $\quad \exists s \in \mathbb{R}^m_+ : \; q_0(x) - \sum_{i=1}^{m} s_i q_i(x) \geq 0, \; \forall x \in \mathbb{R}^n$

- $(S_2) \implies (S_1)$ is always true.
- **The reverse is in general false.**
- **Under which condition (s)** $(S_1) \implies (S_2)$ **?**

The Basic S-Lemma –(Yakubovitch-61, 73)

S-Lemma Let $m = 1$ and suppose

$$\exists \hat{x} \text{ such that } \quad q_1(\hat{x}) > 0.$$

Then, $(S_1) \iff (S_2)$.

$$(S_2) \iff \exists s \in \mathbb{R}_+ : \begin{pmatrix} Q_0 & b_0 \\ b_0^T & c_0 \end{pmatrix} - s \begin{pmatrix} Q_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} \succeq 0$$

An Instrumental Tool

- In Control Theory
- In LMI/SDP reformulations/analysis of QP
- In Robust Optimization Analysis

Extension for $m > 1$??

Even for $m = 2$ with q_i quadratic forms
$q_i(x) = x^T Q_i x, \ Q_i \in S^n, \ i \in [0, 2]$ in general **false**. Need additional assumptions.

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Even for $m = 2$ with q_i quadratic forms
$q_i(x) = x^T Q_i x,\ Q_i \in S^n,\ i \in [0, 2]$ in general **false**. Need additional assumptions.

Theorem–(Polyak 98) Let $m = 2$. Suppose $n \geq 3$ and

- $\exists\ \mu \in \mathbb{R}^2 :\ \mu_1 Q_1 + \mu_2 Q_2 \succ 0$
- $\exists \hat{x}$ such that $q_1(\hat{x}) > 0,\ q_2(\hat{x}) > 0.$

Then, $(S_1) \iff (S_2)$ where

(S_1) $$x^T Q_0 x \geq 0 \quad \forall x \in \mathcal{F} = \{x : q_1(x) \geq 0, q_2(x) \geq 0\}$$

(S_2) $$\exists s \in \mathbb{R}^2_+ :\ Q_0 - \sum_{i=1}^{2} s_i Q_i \succeq 0.$$

Proof. Brickman's Theorem + apply *"Separation of convex sets"*. $\square$

The S-Procedure, Duality and Images under QM

Data: Quadratic Functions $q_i(x) = x^T Q_i x + 2b_i^T x + c_i, \quad Q_i \in S^n$

$$(P) \qquad \inf\{q_0(x) : q_i(x) \geq 0 \; i \in [1, m]\}$$

$$(D) \qquad \sup_{\lambda \in \mathbb{R}^m_+} \inf_x \{q_0(x) - \sum_{i=1}^m \lambda_i q_i(x)\}$$

The S-procedure and Duality are **not** equivalent... But one can derive "simple" connections. The main tool is again the map

$$\psi : \mathbb{R}^n \to \mathbb{R}^{1+m} \;\; \psi(x) = (q_0(x), q_1(x), \ldots, q_m(x))^T$$

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$$\psi : \mathbb{R}^n \to \mathbb{R}^{1+m} \;\; \psi(x) = (q_0(x), q_1(x), \ldots, q_m(x))^T$$

Proposition 1 Suppose $\exists \hat{x}$ such that $q_i(\hat{x}) > 0, \; \forall i = 1, \ldots, m$. Then $\psi(\mathbb{R}^n)$ convex $\implies \{[S_1] \iff [S_2]\}$.

Proposition 2 If $v(P) = v(D)$ **and** $v(D)$ attained, then $\{(S_1) \iff (S_2)\}$.

Note: Both results valid for any functions.

Extension of the Hausdorff-Toeplitz Theorem

Back to images under a quadratic map.

Question: For what values of m is the following claim valid?

Let $A_1, \ldots, A_m$ be $n \times n$ Hermitian matrices. Then the set

$$\{(z^* A_1 z, \ldots, z^* A_m z) : \|z\| = 1, z \in \mathbb{C}^n\}$$

is closed and convex.

- True for $m = 2$ (This is Hausdorff-Toeplitz Theorem)
- Is it true for m = 3???

Answers...

From P. Halmos, "A Hilbert Space Problem Book", 1967:

"It is a pity that it is so very false. It is false for m = 3 and dimension 2; counterexamples are easy to come by."

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Well...Don't trust anyone..!!..

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- **The Hausdorff-Toeplitz theorem is valid for $m = 3$ and $n \geq 2$.**
- Proven by Au-Yeung and Poon, (1979), Binding (1985), Lyubich and Markus (1997).

Some Matrix Related Results

Finsler's Theorem (1936). Let $A, B \in S^n$.
Result 1

$$(F_1)\ \ d^T A d > 0,\ \forall 0 \neq d \in Q_B := \{d : d^T Q d = 0\}$$
$$(F_1) \implies \exists \mu \in \mathbb{R} :\ A + \mu B \succ 0\ \ (\text{trivial} \impliedby)$$

Result 2 Suppose $n \geq 3$. Then,

$$\{x : x^T A x = 0, x^T B x = 0\} = \{0\} \Leftrightarrow \exists \mu \in \mathbb{R}^2 : \mu_1 A + \mu_2 B \succ 0.$$

$RHS \implies A, B$ simultaneously diagonalizable

Once again for 3 and more symmetric matrices ....???......

Theorem (Chen-Yuan (99))
If $\max\{x^T A_1 x, x^T A_2 x, x^T A_3 x\} \geq 0\ \forall x \in \mathbb{R}^n$, then

$$\exists \mu \in \mathbb{R}^3_+\ \sum_{i=1}^{3} \mu_i = 1 \text{ s.t. } \sum_{i=1}^{3} \mu_i A_i \text{ has at most 1 negative eigenvalue.}$$

Convexity of the image of $q(\mathbb{R}^n)$ beyond $m \geq 3$

Let $q : \mathbb{R}^n \to \mathbb{R}^m$ be the quadratic map defined via:

$$q(x) = (q_1(x), \ldots, q_m(x));\ q_i(x) = x^T Q_i x,\ S^n_{++} \ni Q_i,\ i = 1, \ldots, m.$$

The image $q(\mathbb{R}^n) \subset \mathbb{R}^m$ is always convex for $m = 2$, and for $m = 3$ if of some linear combination of Q_1, Q_2, Q_3 is positive definite.

How "close" is the image of $q(\mathbb{R}^n)$ from its convex hull conv $q(\mathbb{R}^n)$?

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How "close" is the image of $q(\mathbb{R}^n)$ from its convex hull $\operatorname{conv} q(\mathbb{R}^n)$?

Theorem (Barvinok (2014) The relative entropy distance from the convex hull of the image of q to the image of q is bounded above by an absolute constant. More precisely, for every $u \in \operatorname{conv} q(\mathbb{R}^n), \; u_1 + \ldots + u_m = 1$, there exists $v \in q(\mathbb{R}^n), \; v_1 + \ldots v_m = 1$ such that

$$D_{KL}(u, v) := \sum_{i=1}^{m} u_i \ln \left(\frac{u_i}{v_i} \right) \leq \tau, \text{ for some absolute ct. } \tau > 0.$$

Replacing $q(\mathbb{R}^n)$ by its convex hull leads to a "constant" loss of information...

Summary

Nonconvex quadratic optimization remains a challenging area, and a source of interesting mathematical questions/problems.

- **Failure of S-procedure for $m \geq 3 \simeq$ Intractability of QP...**
- **Convexity of $q(\mathbb{R}^n) \simeq$ Strong Duality and Computational Tractability...**
- **Identifying more classes of tractable hidden convex problems?**

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Thank you for "Zooming" !