Hidden Convexity in Nonconvex Quadratic Optimization

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Nonconvex Quadratic Optimization

Data: \([Q_i \in S_n, \ b_i \in \mathbb{R}^n, \ c_i \in \mathbb{R}], \ i \in [0, m]\)

\[
\begin{align*}
\inf \ x^T Q_0 x + 2b_0^T x \\
x^T Q_i x + 2b_i^T x + c_i \leq 0, \ i \in [1, p] \\
x^T Q_i x + 2b_i^T x + c_i = 0, \ i \in [p + 1, m], \ x \in \mathbb{R}^n
\end{align*}
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Arise in many and disparate contexts........

- Ill-posed problems/Regularization/Least Squares
- Eigenvalue perturbations
- Optimization algorithms: Trust Region Methods
- Polynomial Optimization problems
- Models for fundamentals Combinatorial/Graph Optimization problems (Max cut, stability number, max clique, etc..)
- Robust Optimization
- Spin Glasses....and much more....!
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**A BRIDGE between: Continuous and Discrete Optimization....**

Thus, not surprisingly so "HARD" to analyze/solve.
Nonconvex optimization are generally non-tractable (NP hard).
However, some classes of nonconvex problems can be solved.

**Hidden convex** problems are nonconvex problems which admit an equivalent convex reformulation.

Focus on **detecting ”Hidden Convexity”** in Nonconvex QP

- Duality
- Lifting and Semidefinite Relaxation
- Exact solutions for some classes of QP
- Convexity of the Image of a Quadratic Map
- The S-Procedure
Duality: A Quick Review

Data \([f, C]\): \(f : \mathbb{R}^n \rightarrow (-\infty, +\infty], \ C \subseteq \mathbb{R}^n\)

Primal Problem

\[
(P) \quad v(P) = \inf \{ f(x) : x \in C \} \equiv \inf_{x \in \mathbb{R}^n} \{ f(x) + \delta_C(x) \}
\]

Dual Problem Uses the same data

\[
(D) \quad v(D) = \sup_y \{-f^*(y) - \delta_C^*(-y) : y \in \text{dom } f^* \cap \text{dom } \delta_C^* \};
\]

with \(f^*(y) := \sup_x \{ \langle x, y \rangle - f(x) \}; \quad \delta_C := \text{indicator of } C.\)

Properties of \((P)-(D)\)

- Dual is always convex (sup-concave)
- Weak duality holds: \(v(P) \geq v(D)\) for any feasible pair \((P)-(D)\)
Duality: Key Questions

\[ \nu(P) = \inf \{ f(x) : x \in C \}; \quad \nu(D) = \sup_y \{-f^*(y) - \delta^*_C(-y)\} \]

- **Zero Duality Gap:** when \( \nu(P) = \nu(D) \)?
- **Strong Duality:** when inf / sup attained?
- **Structure/Relations of Primal-Dual Optimal Sets/Solutions**

  Convex \([f, C]\) + some Regularity Cond. deliver the answers
Less Popular: The Bidual

\[(P) \quad \nu(P) = \inf \{ f(x) : x \in C \} ; \quad (D) \quad \nu(D) = \sup_y \{-f^*(y) - \delta^*_C(-y)\}\]

The dual \((DD)\) of \((D)\) is then in term of the \textbf{bi-conjugate}:

\[(DD) \quad \nu(DD) = \inf_z \{ f^{**}(z) + \delta^{**}_C(z) \}\]

The dual \((D)\) being \textit{always convex}, one has (modulo some Reg. Cond.)

\[\nu(P) \geq \nu(D) = \nu(DD)\]
Less Popular: The Bidual

\[(P) \quad v(P) = \inf \{ f(x) : x \in C \}; \quad (D) \quad v(D) = \sup_y \{-f^*(y) - \delta_C^*(-y)\}\]

The dual \((DD)\) of \((D)\) is then in term of the **bi-conjugate**:

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The dual \((D)\) being **always convex**, one has (modulo some Reg. Cond.)

\[v(P) \geq v(D) = v(DD)\]

- \(v(DD)\) is another lower bound for \(v(P)\)
- \(v(DD)\) natural **convexification** of \(v(P)\) \(\iff f^{**} \leq f\)
- \(v(DD)\) often reveals **hidden convexity** – (or lack of) in \((P)\).
A Prototype in CO: Trust Region is Hidden Convex

The nonconvex trust region suproblem \([Q \in S_n, g \in \mathbb{R}^n, r > 0]\):

\[
(\text{TR}) \quad \text{minimize} \quad \{z^T Qz - 2g^T z : \|z\| \leq r \} \quad z \in \mathbb{R}^n
\]

\(Q \in S_n\) can be diagonalized, i.e., \(\exists\) an orthogonal \(C\)

\[C^T QC = D := \text{diag}(d_1, \ldots, d_n), \quad d_j \in \mathbb{R}, \quad j = 1, \ldots, n; \quad c := Cg\]

Theorem (Ben-Tal and T. (1996)). The nonconvex (TR) is equivalent to the convex problem

\[
(\text{CTR}) \quad \min \left\{ \sum_{j=1}^{n} d_j y_j - 2|c_j| \sqrt{y_j} : \sum_{j=1}^{n} y_j \leq r, \; y \in \mathbb{R}^n_+ \right\}
\]

More precisely, \(\exists y^*\) of (CTR), and corresponding optimal solution of (TR) given by \(z^* = Cx^*\), \(x_j^* = \text{sgnc}_j \sqrt{y_j^*}\), \(\forall j\) and \(\inf(\text{TR}) = \min(\text{CTR})\).

Proof. See more general results (e.g., min. of indefinite quadratic subject to 2-sided indefinite constraints; min of concave quadratic over finitely many convex quadratic) proven via **biduality** in (Ben Tal-Teboulle (96)).
A Prototype in DO: The Max-Cut Problem

Data Input: A graph $G = (V, E)$, $V = \{1, 2, \ldots, n\}$ with weights $w_{ij} = w_{ji} \geq 0$ on the edges $(i, j) \in E$ and with $w_{ij} = 0$ if $(i, j) \notin E$.

Problem: Find the set of vertices $S \subset V$ that maximizes the weight of the edges with one end point in $S$ and the other in its complement $\bar{S}$, i.e., to maximize the total weight across the cut $(S, \bar{S})$. 

Max Cut as a Nonconvex QP

\[
\text{(MC)} \quad \max \left\{ \sum_{i < j} w_{ij} - x_i x_j : x_i^2 = 1, i = 1, \ldots, n \right\}
\]

Can be reformulated equivalently as

\[
\text{(MC)} \quad \max \left\{ x^T Q x : x_i^2 = 1, i = 1, \ldots, n \right\}
\]

where $S_n \ni Q := L^4 \succeq 0, q_{ij} \leq 0 \forall i \neq j$; $L \equiv \text{diag}(W) - W$. 

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### Max Cut as a Nonconvex QP

$$(MC) \quad \max \left\{ \sum_{i < j} w_{ij} \frac{1 - x_i x_j}{2} : x_i^2 = 1, \ i = 1, \ldots, n \right\}$$

Can be reformulated equivalently as

$$(MC) \quad \max \{ x^T Q x : x_i^2 = 1, \ i = 1, \ldots, n \}$$

where $S_n \ni Q := \frac{L}{4} \succeq 0$, $q_{ij} \leq 0 \ \forall i \neq j$; $L \equiv \text{diag } We - W$; $e = (1, \ldots, n)^T$.
Dual Representations of MC–(Shor–87)

\[(MC) \quad \max\{x^T Q x : x_i^2 = 1, \ i = 1, \ldots, n\}\]

DUALITY IS "VERY FLEXIBLE"...
DUAL REPRESENTATIONS OF MC–(Shor–87)

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**DUALITY IS "VERY FLEXIBLE"...**
The following are "equal" upper bounds for (MC)

- \[\min_{u \in \mathbb{R}^n} \{u^T e : \text{Diag}(u) \succeq Q\}\]
- \[\min_{u \in \mathbb{R}^n} \{u^T e + n\lambda_{\max}(Q - \text{Diag}(u))\}\]
- \[\min_{u \in \mathbb{R}^n} \{n\lambda_{\max}(Q + \text{Diag}(u)) : u^T e = 0\}\]
- \[\min_{u \in \mathbb{R}^n} \{u^T e : \lambda_{\min}(\text{Diag}(u) - Q) \geq 0\}\]

**Notation:**
For \(u \in \mathbb{R}^n\): \(\text{Diag}(u) := \text{Diag}(u_1, \ldots, u_n)\), Diagonal Matrix.
For \(S^n \ni Z\), \(\text{diag}(Z) = (Z_{11}, \ldots, Z_{nn})^T \in \mathbb{R}^n\).
One More Dual Bound...The Bidual

\[(MC) \quad \max \{x^T Q x : x_i^2 = 1, \ i = 1, \ldots, n\}\]

Using the (first) previous dual representation:

\[(DMC) \quad \min_{u \in \mathbb{R}^n} \{u^T e : \text{Diag}(u) \succeq Q\}\]

Take the dual of the above dual – The bidual:

\[(R) \quad \max_{Z \in \mathbb{S}^n} \{\text{tr} Q Z : \text{diag}(Z) = e, \ Z \succeq 0\}\]

Here one has: \(\nu(MC) \leq \nu(DMC) = \nu(R)\)

The pair of convex problems \((DMC)-(R)\) are "Semidefinite Optimization problems"
Semi-Definite Programming–SDP Relaxation

$$\min_{x \in \mathbb{R}^m} \{ c^T x : A(x) \succeq 0 \}; \quad \max_{Z \in S^n} \{ \text{tr } A_0 Z : \text{tr } A_i Z = c_i, \ i \in [1, m] \ Z \succeq 0 \}$$

where $A(x) := A_0 + \sum_{i=1}^m x_i A_i$

- SDP are special classes of **convex optimization** problems
- Computationally tractable: Can be **approximately solved** to a desired accuracy in polynomial time
- Naturally occurs in Relaxation of QP via **Lifting**
Semi-Definite Programming–SDP Relaxation

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\begin{align*}
\min_{x \in \mathbb{R}^m} \{ c^T x : A(x) \succeq 0 \}; & \quad \max_{Z \in \mathbb{S}^n} \{ \text{tr} A_0 Z : \text{tr} A_i Z = c_i, \ i \in [1, m] \} Z \succeq 0 \\
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x^T Q x \equiv \text{tr}(Qxx^T). \ \text{Set:} \ X = xx^T. \ \text{Then} \ (MC) \ \text{can be reformulated as}
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(\text{MC}) \quad \max_{X \in S^n} \{ \text{tr} Q X : \text{diag}(X) = e, \text{rank}(X) = 1; X \succeq 0 \}
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- SDP are special classes of \textit{convex optimization} problems
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(MC) \quad \max_{X \in \mathbb{S}^n} \{ \text{tr} Q X : \text{diag}(X) = e, \text{rank}(X) = 1; X \succeq 0 \}
\]

\textbf{DROP the Hard RANK ONE constraint=SDP RELAXATION}

\textbf{(R) \equiv BIDUAL OF MC \equiv SDP RELAXATION}
A Fundamental Question in Nonconvex QP:

Tightness of the SDP relaxation (≡ Bidual Bound for QP)?
A Fundamental Question in Nonconvex QP:

Tightness of the SDP relaxation ($\equiv$ Bidual Bound for QP)?

A General Class of Nonconvex QP

\[(QP) \quad \nu(QP) := \max\{x^T Q x : x^2 \in \mathcal{F}\}\]

\[x^2 \equiv (x^2_1, \ldots, x^2_n)^T; \quad Q \in S_n; \quad \mathcal{F} \subseteq \mathbb{R}^n, \text{ closed convex}\]

Extends the special case: [Max-Cut when $\mathcal{F} = \{e\}]$. 
A Trigonometric Representation of QP

**Notation:** \( \forall X \in S_n, \ \text{arcsin} \ X := \text{arcsin}(X_{ij}); \langle A, B \rangle = \text{tr}(AB) \)

\[ \text{diag}(X) := (X_{11}, \ldots, X_{nn})^T \in \mathbb{R}^n, D := \text{Diag}(d_1, \ldots, d_n) \text{ diag. matrix.} \]
A Trigonometric Representation of QP

Notation: \( \forall X \in S_n, \ \arcsin X := \arcsin(X_{ij}); \langle A, B \rangle = \text{tr}(AB) \)
\( \text{diag}(X) := (X_{11}, \ldots, X_{nn})^T \in \mathbb{R}^n, \ D := \text{Diag}(d_1, \ldots, d_n) \) diag. matrix.

\[
(TQP) \max_{(d,X)} \left\{ \frac{2}{\pi} \langle Q, D \arcsin(X)D \rangle : d \in \mathbb{R}^n_+, \ d^2 \in \mathcal{F}, \ X \succeq 0, \ \text{diag}(X) = e \right\}
\]

Theorem (Goemans-Williamson 95, Nesterov 97, Ye 99)

\[ v(QP) = v(TQP) \]

(TQP) is the key tool to derive \((0, 1) \ni \rho\)-approximate solutions to (QP) \([\rho = .878\ldots \text{ for (MC): } \rho = 2^{-1} \pi \text{ for } Q \succeq 0]\)

\[
\rho v(R) \leq v(TQP) = v(QP) \leq v(R)
\]

where (R) is a semidefinite relaxation of (QP) given by

\[
(R) \max \{ \langle Q, Z \rangle : \ \text{diag}(Z) \in \mathcal{F}, S_n \ni Z \succeq 0 \}.
\]
Nonconvex QP with Exact Solutions

QP with Exact Solutions through their Convex relaxation counterpart?

- \((QP)\) \(v(QP) := \max\{x^T Q x : x^2 \in \mathcal{F}\}\)
- \((R)\) \(\max\{\langle Q, Z \rangle : \text{diag}(Z) \in \mathcal{F}, S^n \ni Z \succeq 0\}\)

Are there \((QP)\) for which \(v(R) = V(QP)\)?

Theorem (Zhang 2000) Let \(S^n \ni Q\) with \(q_{ij} \geq 0\) \(\forall i \neq j\). Then,

\[v(QP) = v(TQP) = v(R)\]

If \(Z\) solves \((R)\) then \(\sqrt{\text{diag}(Z)}\) solves \((QP)\).

Proof based on the key \((TQP)\) representation + some other approximation and penalty arguments.

We will show that this can be proved directly via duality.
Nonconvex QP with Exact Solutions

QP with Exact Solutions through their Convex relaxation counterpart?

- (QP) \( \nu(QP) := \max\{x^T Q x : x^2 \in F\} \)
- (R) \( \max\{\langle Q, Z \rangle : \text{diag}(Z) \in F, S^n \ni Z \succeq 0\} \)

Are there (QP) for which \( \nu(R) = \nu(QP) \)?

**Theorem** (Zhang 2000) Let \( S^n \ni Q \) with \( q_{ij} \geq 0 \forall \ i \neq j \). Then,

\[
\nu(QP) = \nu(TQP) = \nu(R)
\]

If \( Z \) solves (R) then \( \sqrt{\text{diag} Z} \) solves (QP).

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QP with Exact Solutions through their Convex relaxation counterpart?

- $(QP)$ \( v(QP) := \max\{x^T Q x : x^2 \in \mathcal{F}\}$
- $(R)$ \( \max\{\langle Q, Z \rangle : \text{diag}(Z) \in \mathcal{F}, S^n \ni Z \succeq 0\}$

Are there $(QP)$ for which $v(R)=V(QP)$?

**Theorem** (Zhang 2000) Let $S^n \ni Q$ with $q_{ij} \geq 0 \, \forall \, i \neq j$. Then,

\[
v(QP) = v(TQP) = v(R)
\]

If $Z$ solves $(R)$ then $\sqrt{\text{diag} Z}$ solves $(QP)$.

Proof based on the key $(TQP)$ representation + some other approximation and penalty arguments.

We will show that this can be proved directly via duality.
A Bidual Approach to Exact Solutions

\[
(QP) \quad \max\{x^T Q x : x^2 \in \mathcal{F}\} \iff \max_{x,y}\{x^T Q x : y = x^2, y \in \mathcal{F}\}
\]

• A dual of (QP) is (DQP)

\[
(DQP) \quad \min_{u \in \mathbb{R}^n}\left\{ \max_{x \in \mathbb{R}^n} x^T (Q - U) x + \max_{y \in \mathcal{F}} \langle y, u \rangle \right\} = \min_{u \in \mathbb{R}^n}\{\sigma_{\mathcal{F}}(u) : Q - U \preceq 0\}
\]

where \( U := \text{Diag}(u_1, \ldots, u_n) \), \( \sigma_{\mathcal{F}}(u) =: \max\{\langle u, y \rangle : y \in \mathcal{F}\} \equiv \delta^*_\mathcal{F}(u) \).
A Bidual Approach to Exact Solutions

\[(QP) \quad \max\{x^T Qx : x^2 \in \mathcal{F}\} \iff \max_{x,y} \{x^T Qx : y = x^2, y \in \mathcal{F}\}\]

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where \(U := \text{Diag}(u_1, \ldots, u_n)\), \(\sigma_{\mathcal{F}}(u) =: \max\{\langle u, y \rangle : y \in \mathcal{F}\} \equiv \delta^*_F(u)\).

• Bidual: Dual of the dual DQP

\[(D^2QP) \quad \max_{Z \succeq 0} \{\langle Q, Z \rangle + \min_{u \in \mathbb{R}^n} \{\delta^*_F(u) - \langle Z, U \rangle\} \} = \max_{Z \succeq 0} \{\langle Q, Z \rangle : \text{diag}(Z) \in \mathcal{F}\}\]

and \((D^2QP)\) is nothing else but (R).

• Regularity Cond. holds for (DQP) \(\implies \nu(DQP) = \nu(D^2QP)\)
A Simple Duality Proof (Pinar-T. (06))

- **Weak duality:** \( v(QP) \leq v(DQP) \) for any feasible P-D.
- **By Regularity:** \( v(DQP) = v(D^2QP) \), thus \( v(QP) \leq v(D^2QP) \).
- **Ask for equality, i.e. when** \( v(QP) \geq v(D^2QP) \)?
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For any \( Z \) optimal of \( (D^2 QP) \); \( x = \sqrt{\text{diag}(Z)} \) feas. for \( (QP) \) since

\[
\text{diag}(Z) \in \mathcal{F} \implies x^2 = \text{diag}(Z) \in \mathcal{F}.
\]
A Simple Duality Proof (Pinar-T. (06))

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Thus, with $x$ feasible for (QP) and $Z$ optimal for $(D^2QP)$ we have:

$$v(QP) \geq \sum_{i,j} q_{ij}x_ix_j = \sum_{i} q_{ii}Z_{ii} + 2\sum_{i<j} q_{ij}\sqrt{Z_{ii}Z_{jj}}, \text{ while}$$

$$v(D^2QP) = \sum_{i,j} q_{ij}Z_{ij} = \sum_{i} q_{ii}Z_{ii} + 2\sum_{i<j} q_{ij}Z_{ij},$$

$$\implies v(QP) - v(D^2QP) \geq 0 \iff \sum_{i \neq j} q_{ij}(\sqrt{Z_{ii}Z_{jj}} - Z_{ij}) \geq 0$$
A Simple Duality Proof (Pinar-T. (06))

- **Weak duality:** $v(QP) \leq v(DQP)$ for any feasible P-D.
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Thus, with $x$ feasible for (QP) and $Z$ optimal for $(D^2 QP)$ we have:

$$v(QP) \geq \sum_{i,j} q_{ij}x_i x_j = \sum_i q_{ii}Z_{ii} + 2 \sum_{i < j} q_{ij} \sqrt{Z_{ii}Z_{jj}}, \text{ while}$$

$$v(D^2 QP) = \sum_{i,j} q_{ij}Z_{ij} = \sum_i q_{ii}Z_{ii} + 2 \sum_{i < j} q_{ij}Z_{ij},$$

$$\Rightarrow \quad v(QP) - v(D^2 QP) \geq 0 \iff \sum_{i \neq j} q_{ij}(\sqrt{Z_{ii}Z_{jj}} - Z_{ij}) \geq 0$$

**But** $Z \succeq 0 \Rightarrow Z_{ii} \geq 0$, $Z_{ii}Z_{jj} \geq Z_{ij}^2 \forall i \neq j$

Therefore since we assumed $q_{ij} \geq 0$, $\forall i \neq j$ we are done!
Quadratic Maps with Convex Images

An old and classical subject in Mathematics....

In the *complex space* goes back to:

**Hausdorff-Toeplitz Theorem [1918]:** The numerical range of a linear operator is closed and convex.

Explicitly, in finite dimension, with $A, B, n \times n$, Hermitian matrices one has: $\{(z^*Az, z^*Bz) : \|z\| = 1, z \in \mathbb{C}^n\} \subseteq \mathbb{R}^2$, is closed convex.
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**The Real Case – Let** $A, B \in S^n$.

**Dines Theorem (1940)** $\{(x^TAx, x^TBx) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$ is convex.
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The Real Case – Let $A, B \in S^n$.

Dines Theorem (1940) $\{(x^TAx, x^TBx) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$ is convex.

A Key Result: Brickman’s Theorem (1961). If $n \geq 3$ then

$B := \{(x^TAx, x^TBx) : \|x\| = 1, x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$ is closed convex

Note: Brickman’s Theorem fails for $n = 2$
Let $A_i \in S^n$ (indefinite)

For $m \geq 3$, $q_i(x) = x^T A_i x$, $i \in [1, m]$; **Quadratic forms**

For $q_i(x) = x^T A_i x - 2 b_i^T x + c_i$; **Quadratic functions**

$q : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $q(x) \equiv (q_1(x), \ldots, q_m(x))^T$

**Question:** When is $q(\mathbb{R}^n)$ convex for quadratic forms/functions?
Why do we care about the convexity of \( q(\mathbb{R}^n) \)?

At the root of fundamental questions/answers for "Quadratic Problems"

A. Quadratic Optimization e.g., Duality, Optimality, SDP...
B. Matrix related questions e.g., Simultaneous diag.
C. The S-Procedure
D. Computational Tractability
Why do we care about the convexity of $q(\mathbb{R}^n)$?

At the root of fundamental questions/answers for "Quadratic Problems"

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C. The S-Procedure
D. Computational Tractability

Example: $(Q) \min \{ q_0(x) : q_i(x) \leq 0, \ i = 1, \ldots, m, x \in \mathbb{R}^n \}$.

Let $W := \{(q_0(x), \ldots, q_m(x)) : x \in \mathbb{R}^n \}$.

Then $(Q)$ equivalent to:

$$\min \{ s_0 : s_i \leq 0, \ i = 1, \ldots, m, s \in W \}$$

$(Q)$ is convex iff the image $W$ convex.
Homogeneous Quadratic Forms: \( m=3 \)

Let \( S^n \ni A_i, \quad q_i(x) = x^T A_i x, \quad i = 1, 2, 3. \)
Homogeneous Quadratic Forms: $m=3$

Let $S^n \ni A_i, \quad q_i(x) = x^T A_i x, \quad i = 1, 2, 3.$

**Theorem 1** Suppose $n \geq 3$ and there exists $\mu \in \mathbb{R}^3$ such that

$$\mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3 \succ 0.$$ 

Then $\{(q_1(x), q_2(x), q_3(x)) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^3$ is closed convex.
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Nonhomogeneous Case: $m=2$ Let $q_i$ be quadratic functions, i.e.,

$$q_i(x) = x^T A_i x - 2b_i^T x + c_i, \quad b_i \in \mathbb{R}^n, c_i \in \mathbb{R}, A_i \in S^n, \ i = 1, 2.$$
Quadratic Maps with Convex Images - Polyak (98)

- **Homogeneous Quadratic Forms: m=3**

  Let $S^n \ni A_i$, $q_i(x) = x^T A_i x$, $i = 1, 2, 3$.

  **Theorem 1** Suppose $n \geq 3$ and there exists $\mu \in \mathbb{R}^3$ such that

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  **Theorem 2** Suppose $n \geq 2$ and there exists $\mu \in \mathbb{R}^2$ such that

  $$\mu_1 A_1 + \mu_2 A_2 \succ 0.$$  

  Then $\{(q_1(x), q_2(x)) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$ is closed convex.

The proofs rely on the nontrivial Brickman’s theorem. We will give a simple direct proof of Theorem 2.
Problem Reformulation

Given $q_i(x) = x^T A_i x - 2b_i^T x + c_i, \quad i = 1, 2, \quad q(x) = (q_1(x), q_2(x))^T$

We want to prove that for all $n \geq 2$:

\[(\ast) \exists \mu \in \mathbb{R}^2 \text{ s.t. } \mu_1 A_1 + \mu_2 A_2 \succ 0 \implies q(\mathbb{R}^n) \text{ convex}\]

Under $(\ast)$ we can assume $A_2 \succ 0$. Thus,

$$q(\mathbb{R}^n) := \{(q_1(x), ||x||^2)^T : x \in \mathbb{R}^n\} = \{(s, t) : s = q_1(x), t = ||x||^2\}.$$

For any $n \geq 2$, the sphere $S_t := \{x : ||x||^2 = t\}$ is connected. Thus,$^1$

$$q(\mathbb{R}^n) = \{(s, t) : \inf_{x \in S_t} q_1(x) \leq s \leq \sup_{x \in S_t} q_1(x), \quad t \geq 0\}$$

Therefore $q(\mathbb{R}^n)$ will be convex if we can show that

$$l(t) := \inf_{x \in S_t} q_1(x) \text{ convex in } t \quad (u(t) := \sup_{x \in S_t} q_1(x) \text{ concave in } t)$$

\[\text{Notes:} \quad ^1\text{Let } C \text{ be a connected subset of } \mathbb{R}^n. \text{ Then any real valued function } f \text{ defined and continuous on } \mathbb{R}^n \text{ attains in } C \text{ every value between } \inf_{x \in C} f(x) \text{ and } \sup_{x \in C} f(x).\]
**Lemma** For $t \geq 0$ define the function

$$\ell(t) := \min\{q_1(x) : x \in S_t\} \equiv \min\{x^T A_1 x - 2b_1^T x : x \in S_t\}.$$ 

Then, $\ell(\cdot)$ is a convex function on $\mathbb{R}_+$. 

**Proof.** via biduality !
Direct Proof of Theorem 2

Lemma For \( t \geq 0 \) define the function

\[
\ell(t) := \min\{q_1(x) : x \in S_t\} \equiv \min\{x^T A_1 x - 2b_1^T x : x \in S_t\}.
\]

Then, \( \ell(\cdot) \) is a convex function on \( \mathbb{R}_+ \).

Proof. via biduality !

- Applying the Lemma implies that for all \( n \geq 2 \),

\[
q(\mathbb{R}^n) = \{(s, t) : \inf_{x \in S_t} q_1(x) \leq s \leq \sup_{x \in S_t} q_1(x), \ t \geq 0\}
\]

\[
q(\mathbb{R}^n) = \{(s, t) : \ell(t) \leq s \leq u(t), \ t \geq 0\} \text{ is convex in } \mathbb{R}^2.
\]

- Note: Importance of dimension. For \( n = 1 \), Lemma remains true.. but the set \( (S_t = \{x \in \mathbb{R} : |x| = 1\}) \) is not connected!
A General Nonconvex Class: Ratio of QP

Minimizing ratio of indefinite quadratic functions over an Ellipsoid

\[(RQ) \quad f_* := \inf \left\{ \frac{f_1(x)}{f_2(x)} : \|Lx\|^2 \leq \rho \right\} \]

\[f_i(x) = x^T A_i x + 2b_i^T x + c_i, \quad i = 1, 2\]

\[A_i = A_i^T \in \mathbb{R}^{n \times n}, \quad b_i \in \mathbb{R}^n, \quad c_i \in \mathbb{R}, \quad L \in \mathbb{R}^{r \times n}, \quad \rho > 0\]

The feasible set

\[\mathcal{F} := \{x \in \mathbb{R}^n : \|Lx\|^2 \leq \rho\}\]

represents a (possibly degenerate) ellipsoid.

**Assumption:** Problem (RQ) is well defined, i.e., \(f_2(x) > 0\) for every \(x \in \mathcal{F}\).

**Motivation** Arises in Estimation Problems: Regularized Total LS \(Ax \approx b\) with \((A, b)\) noisy data.

- Regularized Total Least Squares Problem (RTLS):
  \[f_1(x) = \|Ax - b\|^2, \quad f_2(x) = \|x\|^2 + 1\] (both nice convex functions...but ratio is not!)
**Theorem** The minimum of problem (RQ) is attained if either the feasible set is compact – or when \( r < n \) if the following holds:

\[
[SC] \quad \lambda_{\min}(M_1, M_2) < \lambda_{\min}(F^T A_1 F, F^T A_2 F),
\]

where

\[
M_1 = \begin{pmatrix} F^T A_1 F & F^T b_1 \\ b_1^T F & c_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} F^T A_2 F & F^T b_2 \\ b_2^T F & c_2 \end{pmatrix}
\]

\( F \) is an \( n \times (n - r) \) matrix whose columns form an orthonormal basis for the null space of \( L \), and \( \lambda_{\min}(A, B) := \max\{\lambda : A - \lambda B \succeq 0\} \)

♣ The proof relies on asymptotic tools for nonconvex functions.
Attainment of the Minimum for RQ

**Theorem** The minimum of problem (RQ) is attained if either the feasible set is compact – or when $r < n$ if the following holds:

$$[SC] \quad \lambda_{\min}(M_1, M_2) < \lambda_{\min}(F^T A_1 F, F^T A_2 F),$$

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♣ The proof relies on asymptotic tools for nonconvex functions.

[SC] plays a key role for establishing results in two directions:

- ✓ An Exact SDP Relaxation for (RQ).
- Convergence/complexity analysis of a fast algorithm for solving (RQ).
**Theorem**  Let \( n \geq 2 \) and suppose [SC] holds. Then,

\[
\text{val (RQ)} = \text{val}(D), \quad \text{where (D)} \text{ is given by}
\]

\[
\max_{\beta \geq 0, \alpha \in \mathbb{R}} \left\{ \alpha : \begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} \succeq \alpha \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} - \beta \begin{pmatrix} L^T L & 0 \\ 0 & -\rho \end{pmatrix} \right\}
\]

**Proof** relies on strong duality for homogeneous QP with two constraints + the attainability Condition [SC].

- The solution of (RQ) can be extracted from the solution of the semidefinite formulation.
- (RQ) belongs to the **privileged class of Hidden Convex Problems**...
The S-Procedure – (Yakubovitch-61)

\[ q_i(x) = x^T Q_i x + 2b_i^T x + c_i, \quad Q_i \in S^n, \ i = 0, \ldots, m. \]

\[ \mathcal{F} := \{ x \in \mathbb{R}^n : q_i(x) \geq 0, \ i = 1, \ldots, m \} \]

Consider the following statements:

(S1) \quad q_0(x) \geq 0 \quad \forall x \in \mathcal{F}

(S2) \quad \exists s \in \mathbb{R}^m_+ : q_0(x) - \sum_{i=1}^{m} s_i q_i(x) \geq 0, \forall x \in \mathbb{R}^n

- (S2) \implies (S1) is always true.
- **The reverse is in general false.**
- **Under which condition (s) (S1) \implies (S2) ?**
The Basic S-Lemma –(Yakubovitch-61, 73)

**S-Lemma** Let \( m = 1 \) and suppose

\[
\exists \hat{x} \text{ such that } q_1(\hat{x}) > 0.
\]

Then, \((S_1) \iff (S_2)\).

\[
(S_2) \iff \exists s \in \mathbb{R}_+ : \left( \begin{array}{cc} Q_0 & b_0 \\ b_0^T & c_0 \end{array} \right) - s \left( \begin{array}{cc} Q_1 & b_1 \\ b_1^T & c_1 \end{array} \right) \succeq 0
\]

**An Instrumental Tool**
- In Control Theory
- In LMI/SDP reformulations/analysis of QP
- In Robust Optimization Analysis
Extension for $m > 1$?

Even for $m = 2$ with $q_i$ quadratic forms 
$q_i(x) = x^T Q_i x$, $Q_i \in S^n$, $i \in [0, 2]$ in general false. Need additional assumptions.
Even for $m = 2$ with $q_i$ quadratic forms $q_i(x) = x^T Q_i x$, $Q_i \in S^n$, $i \in [0, 2]$ in general false. Need additional assumptions.

**Theorem—(Polyak 98)** Let $m = 2$. Suppose $n \geq 3$ and

- $\exists \mu \in \mathbb{R}^2 : \mu_1 Q_1 + \mu_2 Q_2 \succ 0$
- $\exists \hat{x}$ such that $q_1(\hat{x}) > 0$, $q_2(\hat{x}) > 0$.

Then, $(S_1) \iff (S_2)$ where

$$(S_1) \quad x^T Q_0 x \geq 0 \quad \forall x \in F = \{x : q_1(x) \geq 0, q_2(x) \geq 0\}$$

$$(S_2) \quad \exists s \in \mathbb{R}^2_+ : Q_0 - \sum_{i=1}^{2} s_i Q_i \succeq 0.$$

**Proof.** Brickman’s Theorem + apply "Separation of convex sets". □
The S-Procedure, Duality and Images under QM

Data: Quadratic Functions \( q_i(x) = x^T Q_i x + 2b_i^T x + c_i, \quad Q_i \in S^n \)

\[(P)\quad \inf \{ q_0(x) : q_i(x) \geq 0 \quad i \in [1, m] \} \]

\[(D)\quad \sup_{\lambda \in \mathbb{R}^m_+} \inf_x \{ q_0(x) - \sum_{i=1}^{m} \lambda_i q_i(x) \} \]

The S-procedure and Duality are not equivalent... But one can derive "simple" connections. The main tool is again the map

\[ \psi : \mathbb{R}^n \to \mathbb{R}^{1+m} \quad \psi(x) = (q_0(x), q_1(x), \ldots, q_m(x))^T \]
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\psi : \mathbb{R}^n \to \mathbb{R}^{1+m} \quad \psi(x) = (q_0(x), q_1(x), \ldots, q_m(x))^T
\]

**Proposition 1** Suppose \( \exists \hat{x} \) such that \( q_i(\hat{x}) > 0, \ \forall i = 1, \ldots, m \). Then \( \psi(\mathbb{R}^n) \) convex \( \implies \{ [S_1] \iff [S_2] \} \).

**Proposition 2** If \( \nu(P) = \nu(D) \) and \( \nu(D) \) attained, then \( \{ (S_1) \iff (S_2) \} \).

Note: Both results valid for any functions.
Extension of the Hausdorff-Toeplitz Theorem

Back to images under a quadratic map.

Question: For what values of \( m \) is the following claim valid?

Let \( A_1, \ldots, A_m \) be \( n \times n \) Hermitian matrices. Then the set

\[
\{(z^* A_1 z, \ldots, z^* A_m z) : \|z\| = 1, z \in \mathbb{C}^n\}
\]

is closed and convex.

- True for \( m = 2 \) (This is Hausdorff-Toeplitz Theorem)
- Is it true for \( m = 3 \)???
From P. Halmos, "A Hilbert Space Problem Book", 1967:

“It is a pity that it is so very false. It is false for $m = 3$ and dimension 2; counterexamples are easy to come by.”
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Well...Don’t trust anyone..!!..
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Well...Don’t trust anyone..!!..

- **The Hausdorff-Toeplitz theorem is valid for** \( m = 3 \) **and** \( n \geq 2 \).
Some Matrix Related Results

Finsler’s Theorem (1936). Let $A, B \in S^n$.

Result 1

\[(F_1) \quad d^T Ad > 0, \quad \forall 0 \neq d \in Q_B := \{d : d^T Qd = 0\}\]

\[(F_1) \implies \exists \mu \in \mathbb{R} : A + \mu B \succ 0 \quad \text{(trivial } \iff \text{)}\]

Result 2 Suppose $n \geq 3$. Then,

\[\{x : x^T Ax = 0, x^T Bx = 0\} = \{0\} \iff \exists \mu \in \mathbb{R}^2 : \mu_1 A + \mu_2 B \succ 0.\]

RHS $\implies A, B$ simultaneously diagonalizable

Once again for 3 and more symmetric matrices ....???......

Theorem (Chen-Yuan (99))

If $\max\{x^T A_1 x, x^T A_2 x, x^T A_3 x\} \geq 0 \quad \forall x \in \mathbb{R}^n$, then

\[\exists \mu \in \mathbb{R}^3_+ \sum_{i=1}^{3} \mu_i = 1 \quad \text{s.t.} \quad \sum_{i=1}^{3} \mu_i A_i \text{ has at most 1 negative eigenvalue}.\]
Convexity of the image of $q(\mathbb{R}^n)$ beyond $m \geq 3$

Let $q : \mathbb{R}^n \to \mathbb{R}^m$ be the quadratic map defined via:

$$q(x) = (q_1(x), \ldots, q_m(x)); \quad q_i(x) = x^T Q_i x, \quad S^n_{++} \ni Q_i, \quad i = 1, \ldots, m.$$ 

The image $q(\mathbb{R}^n) \subset \mathbb{R}^m$ is always convex for $m = 2$, and for $m = 3$ if of some linear combination of $Q_1, Q_2, Q_3$ is positive definite.

How "close" is the image of $q(\mathbb{R}^n)$ from its convex hull $\text{conv} \ q(\mathbb{R}^n)$?
Convexity of the image of $q(\mathbb{R}^n)$ beyond $m \geq 3$

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How "close" is the image of $q(\mathbb{R}^n)$ from its convex hull $\text{conv} \ q(\mathbb{R}^n)$?

**Theorem (Barvinok (2014))** The relative entropy distance from the convex hull of the image of $q$ to the image of $q$ is bounded above by an absolute constant. More precisely, for every $u \in \text{conv} \ q(\mathbb{R}^n), \ u_1 + \ldots + u_m = 1$, there exists $\nu \in q(\mathbb{R}^n), \ \nu_1 + \ldots \nu_m = 1$ such that

$$D_{KL}(u, \nu) := \sum_{i=1}^{m} u_i \ln \left( \frac{u_i}{\nu_i} \right) \leq \tau, \text{ for some absolute ct. } \tau > 0.$$ 

Replacing $q(\mathbb{R}^n)$ by its convex hull leads to a "constant" loss of information...
Nonconvex quadratic optimization remains a challenging area, and a source of interesting mathematical questions/problems.

- Failure of $S$-procedure for $m \geq 3 \simeq$ Intractability of QP...
- Convexity of $q(\mathbb{R}^n) \simeq$ Strong Duality and Computational Tractability...
- Identifying more classes of tractable hidden convex problems?
Nonconvex quadratic optimization remains a challenging area, and a source of interesting mathematical questions/problems.

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**Thank you for “Zooming”!**