Progressive Hedging and Asynchronous Projective Hedging for Convex Stochastic Programming

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Typical ADMM and Operator Splitting Applications

- The most prominent applications of operator splitting and ADMM-class algorithms are in machine learning and image processing
- There has not been much operator splitting work on “OR-style” optimization problems
- With one exception:

Stochastic Programming

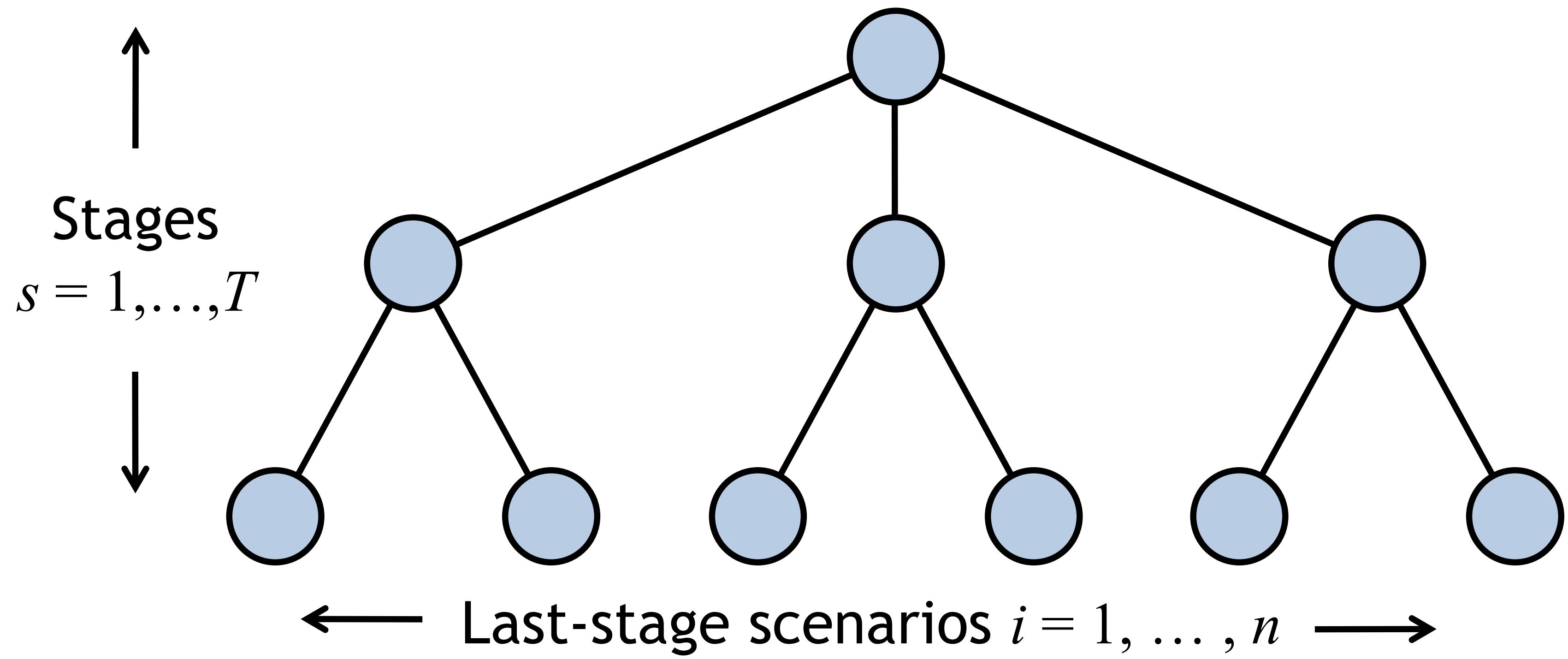
- Solving LP etc. models on an unfolding tree of random future scenarios
- Rockafellar and Wet’s *progressive hedging* algorithm (1991)

Progressive Hedging

- Working paper in late 1987, published in *Mathematics of Operations Research* in 1991
- Rockafellar and Wets knew that their method was a form of DR splitting / ADMM algorithm
- But proved its convergence from first principles
- After all, monotone operators and the ADMM were not much known in the OR community at the time
- I will present the method from an ADMM point of view, then switch to projective splitting

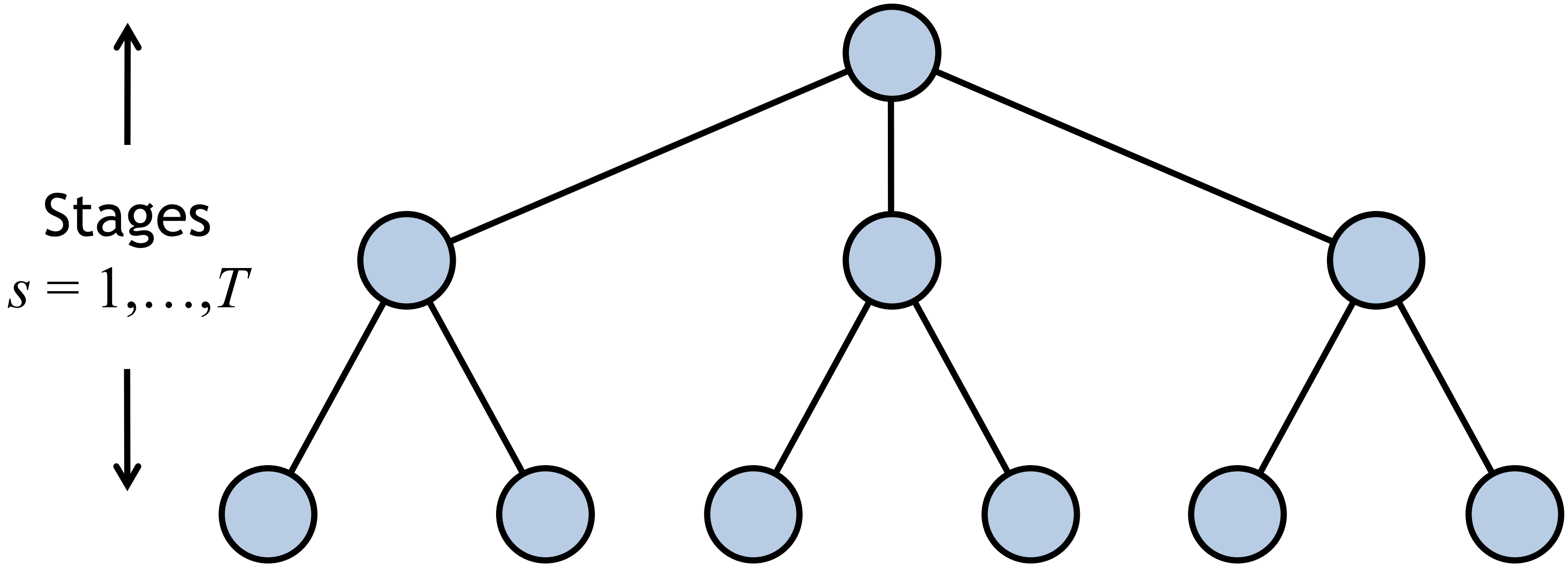
The Scenario Tree

- Consider a standard stochastic programming scenario tree:



- π_i is the probability of last-stage scenario $i = 1, \ldots, n$
- Will use “scenario” as a shorthand for “last-stage scenario”
- Typically a discrete-time and sampled approximation of some infinite or much larger model

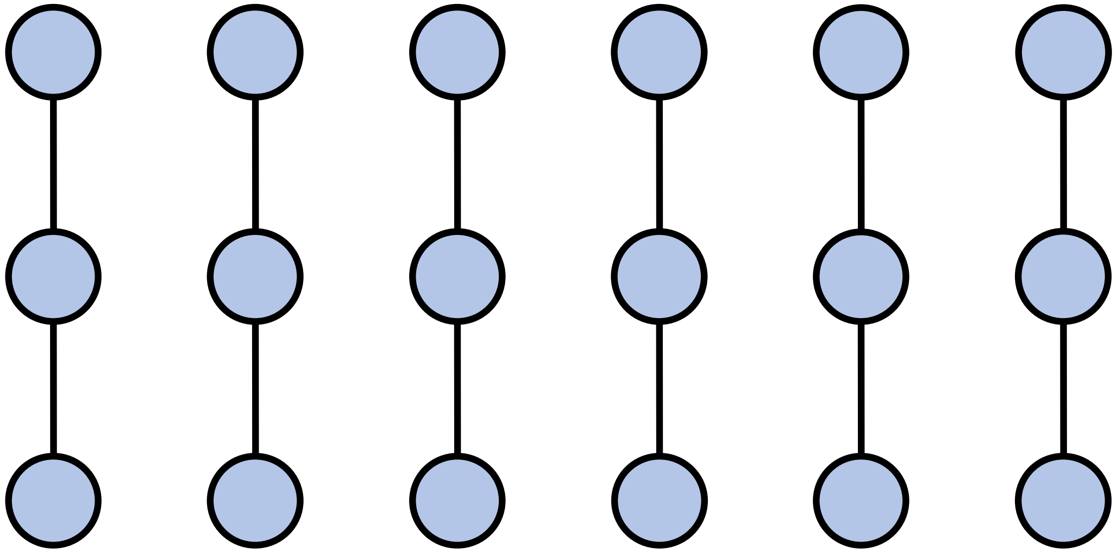
Stochastic Programming



- System walks randomly from the root to some leaf
- At each node there are decision variables, for example
  - How much of an investment to buy or sell
  - How much to run a power generator, etc...
- ... and constraints that depend on earlier decisions
- Model alternates decisions and uncertainty resolution

Problem Formulation and Notation

- Replicate decision variables: n copies at every stage



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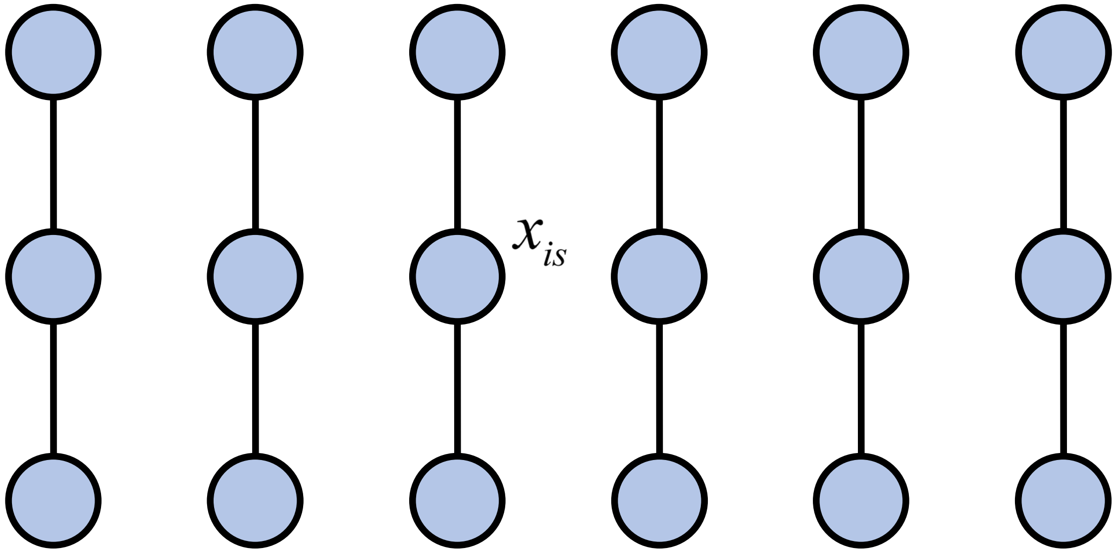

- x_{is} is the vector of decision variables for scenario i at stage s

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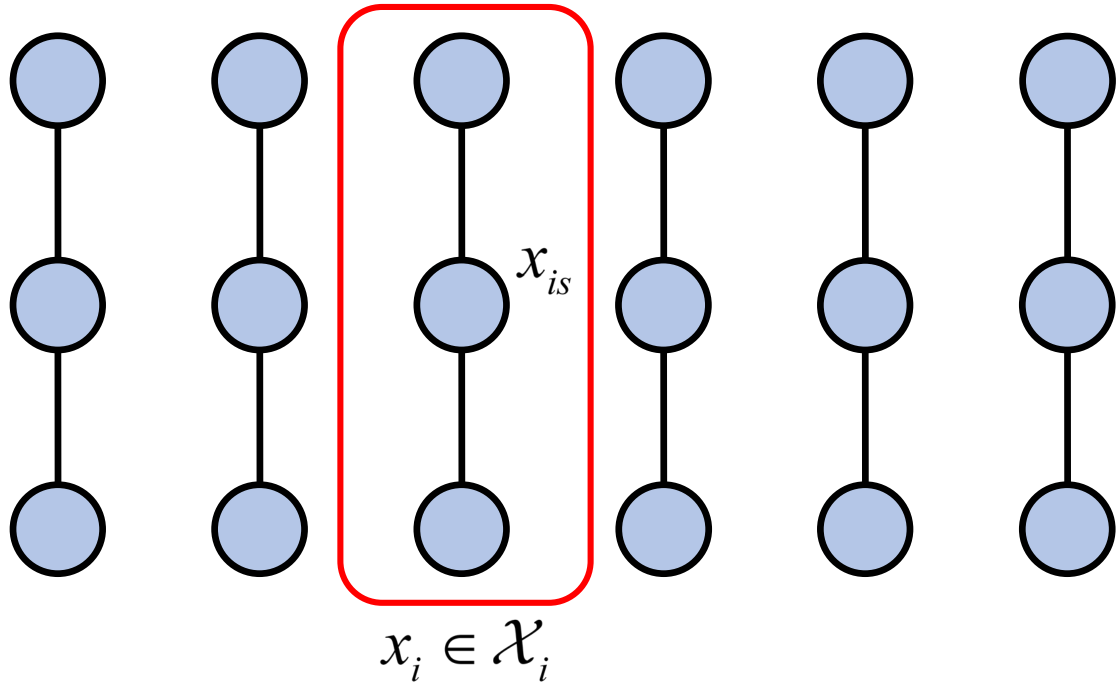

- x_{is} is the vector of decision variables for scenario i at stage s
- $\mathcal{X}_i$ is the space of all variables pertaining to scenario i ; elements are $x_i = (x_{i1}, \ldots, x_{iT})$

Problem Formulation and Notation

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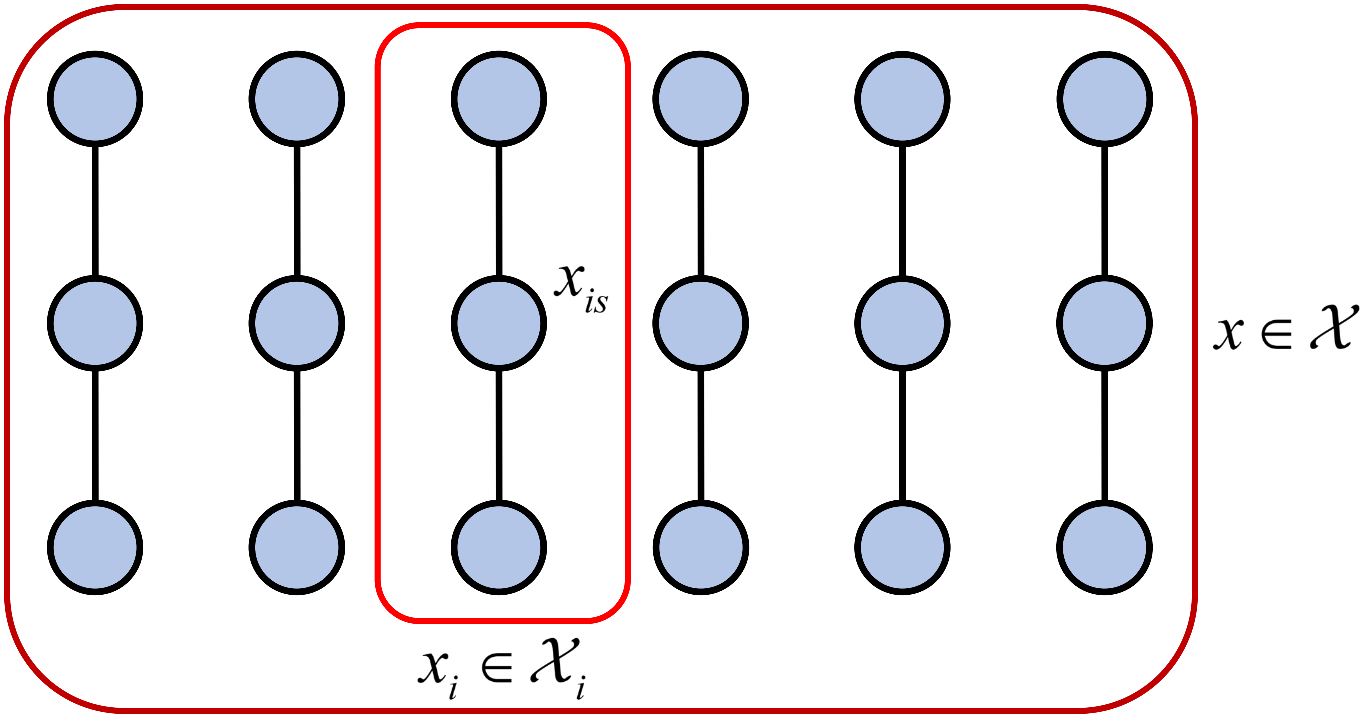
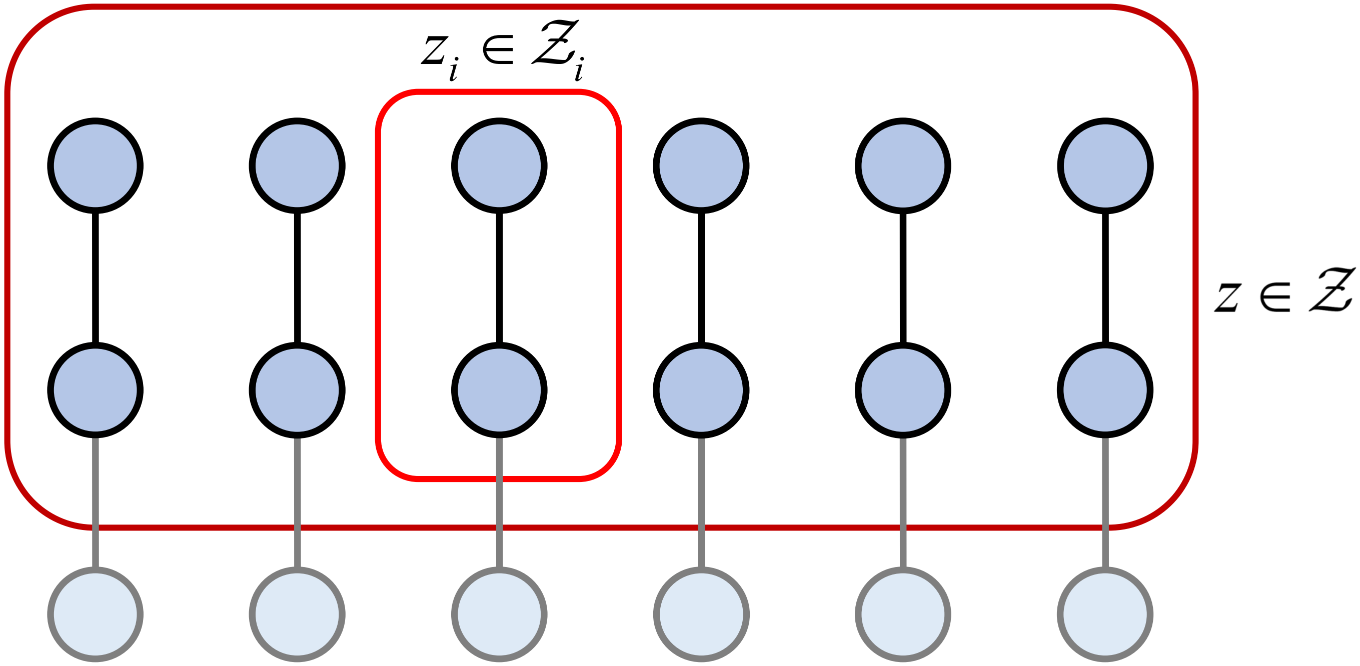


- x_{is} is the vector of decision variables for scenario i at stage s
- $\mathcal{X}_i$ is the space of all variables for scenario i; elements are
  $x_i = (x_{i1}, \ldots, x_{iT})$
- $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is space of all decision variables; elements are
  $x = (x_1, \ldots, x_n) = \big((x_{11}, \ldots, x_{1T}), \ldots, (x_{n1}, \ldots, x_{nT})\big)$

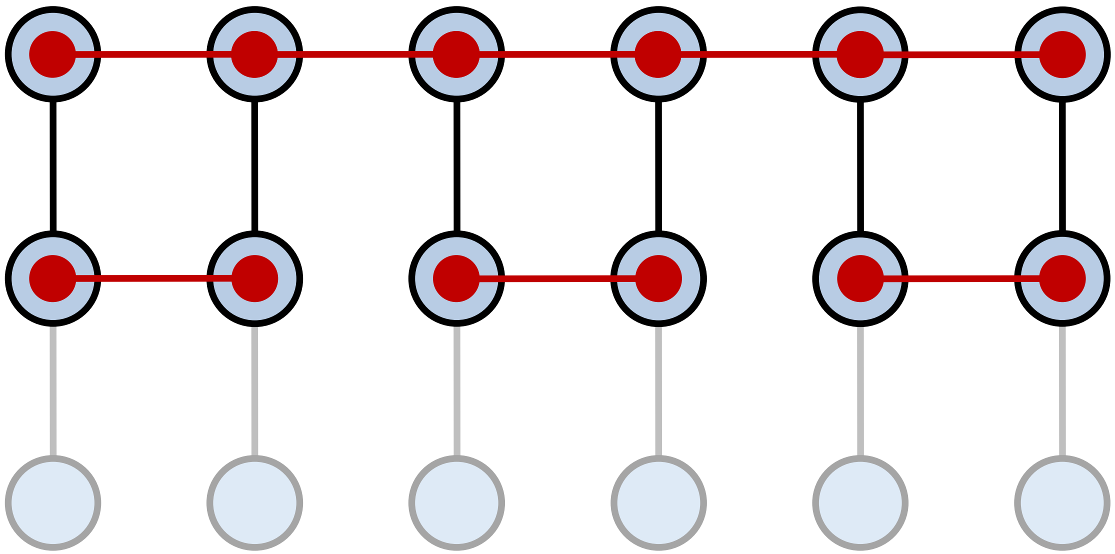
Problem Formulation and Notation

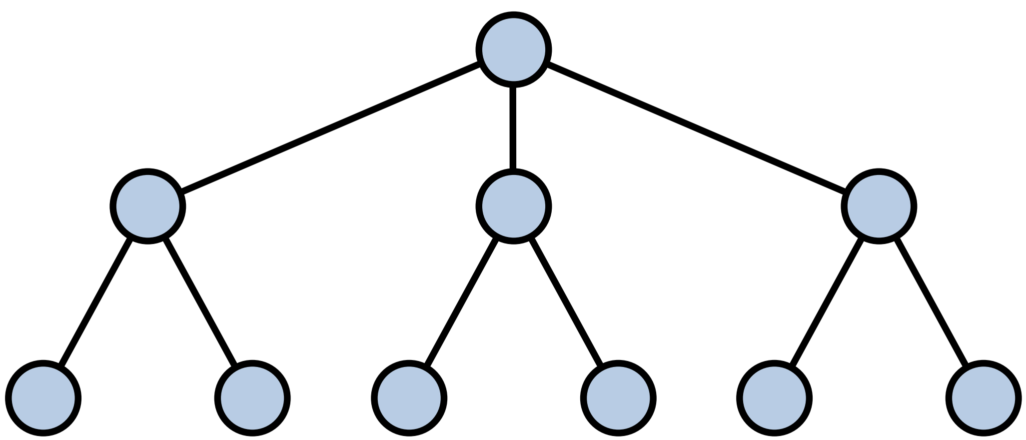


- $\mathcal{Z}_i$ is $\mathcal{X}_i$ without the last stage; elements $z_i = (z_{i1}, \ldots, z_{i,T-1})$
- $\mathcal{Z} = \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n$ is the space of all variables except the last stage: elements $z = (z_1, \ldots, z_n) = \big((z_{11}, \ldots, z_{1,T-1}), \ldots, (z_{n1}, \ldots, z_{n,T-1})\big)$

Nonanticipativity Subspace

- $\mathcal{N} \subset \mathcal{Z}$ is the subspace of $\mathcal{Z}$ meeting the *nonanticipativity constraints* that $z_{is} = z_{js}$ whenever scenarios i and j are indistinguishable at stage s





Projecting onto the Nonanticipativity Space

- Following Rockafeller and Wets (1991), we use the following probability-weighted inner product on $\mathcal{Z}$:

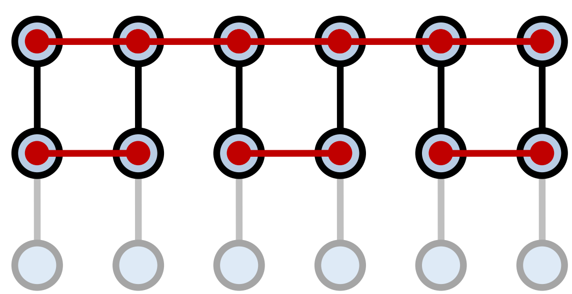
$$\left\langle (z_1,\ldots,z_n),(q_1,\ldots,q_n) \right\rangle = \sum_{i=1}^{n} \pi_i \left\langle z_i, q_i \right\rangle$$

- With this inner product, the projection map $\mathrm{proj}_{\mathcal{N}} : \mathcal{Z} \to \mathcal{N}$ is given by

$$\mathrm{proj}_{\mathcal{N}}(q) = z, \quad \text{where}$$

$$z_{is}^{k+1} = \frac{1}{\left(\sum_{j \in S(i,s)} \pi_j\right)} \sum_{j \in S(i,s)} \pi_j q_{js}^{k+1} \qquad i = 1,\ldots,n, \;\; s = 1,\ldots,T-1$$

and $S(i,s)$ is the set of scenarios indistinguishable from scenario i at time s.



Now Let’s Apply the ADMM

- So far, I have just shown the formulation of Rockafellar & Wets (1991) with some minor notation adjustments
- But now will derive PH using the ADMM instead of first principles

ADMM Notation

- Suppose $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is a convex function
- Suppose $g : \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ is a convex function
- Suppose M is linear map $\mathcal{X} \to \mathcal{Z}$

$$\boxed{\min f(x) + g(Mx)}$$

- Equivalent formulation: $\boxed{\begin{array}{ll} \min & f(x) + g(z) \\ \text{ST} & Mx = z \end{array}}$

- The ADMM, for some fixed constant $\rho > 0$:

$$x^{k+1} \in \operatorname*{Arg\,min}_{x \in \mathbb{R}^n} \left\{ f(x) + \left\langle w^k, Mx \right\rangle + \tfrac{\rho}{2} \left\| Mx - z^k \right\|^2 \right\}$$

$$z^{k+1} \in \operatorname*{Arg\,min}_{z \in \mathbb{R}^m} \left\{ g(z) - \left\langle w^k, z \right\rangle + \tfrac{\rho}{2} \left\| Mx^{k+1} - z \right\|^2 \right\}$$

$$w^{k+1} = w^k + \rho(Mx^{k+1} - z^{k+1})$$

Setting Up the ADMM Formulation for Stochastic Programming

For each $i = 1, \ldots, n$, let

- $f_i : \mathcal{X}_i \to \mathbb{R} \cup \{+\infty\}$ be given by $f_i(x_i) = \pi_i h_i(x_i)$, where $h_i(\cdot)$ is the cost function for scenario i and $+\infty$ if any constraint within scenario i is violated
- $M_i : \mathcal{X}_i \to \mathcal{Z}_i$ be the map that just drops the last-stage variables from scenario i

h_i encapsulates all the costs and constraints across all stages in the (hypothetical) situation that you know the final outcome will be scenario i

Then our stochastic program is equivalent to

$$\boxed{\begin{array}{ll} \min & \displaystyle\sum_{i=1}^{p} f_i(x_i) \\ \text{ST} & (M_1 x_1, \ldots, M_n x_n) \in \mathcal{N} \end{array}}$$

Applying the ADMM

$$f(x)=\sum_{i=1}^{n} f_i(x_i) \qquad g(z)=\begin{cases}0, & z\in\mathcal{N}\\ +\infty, & z\notin\mathcal{N}\end{cases} \qquad M:(x_1,\ldots,x_n)\mapsto(M_1x_1,\ldots,M_nx_n)$$

Then the problem is equivalent to $\min f(x)+g(Mx)$

Applying the ADMM (and thus DR), we obtain

$$\boxed{\begin{aligned} x_i^{k+1} &= \underset{x_i\in\mathcal{X}_i}{\arg\min}\left\{f_i(x_i)+\left\langle M_ix_i,w_i^k\right\rangle+\tfrac{\rho}{2}\left\|M_ix_i-z_i^k\right\|^2\right\} && i=1,\ldots,n\\ z^{k+1} &= \mathrm{proj}_{\mathcal{N}}\left(Mx_1^{k+1}\right) && \\ w^{k+1} &= w^k+\rho(Mx^{k+1}-z^{k+1}) && i=1,\ldots,n \end{aligned}}$$

- Note that we always have $w^k=(w_1^k,\ldots,w_p^k)\in\mathcal{N}^\perp$. Why?
  - Projection means that $Mx-z\in\mathcal{N}^\perp$
  - Or, note that in ADMM/DR we always have $w^k\in\partial g(z^k)$
- Aside: applying DR to subspace indicator functions like g is equivalent to Spingarn's method of partial inverses

Progressive Hedging

- Writing the z and w operations out in detail, we obtain PH:

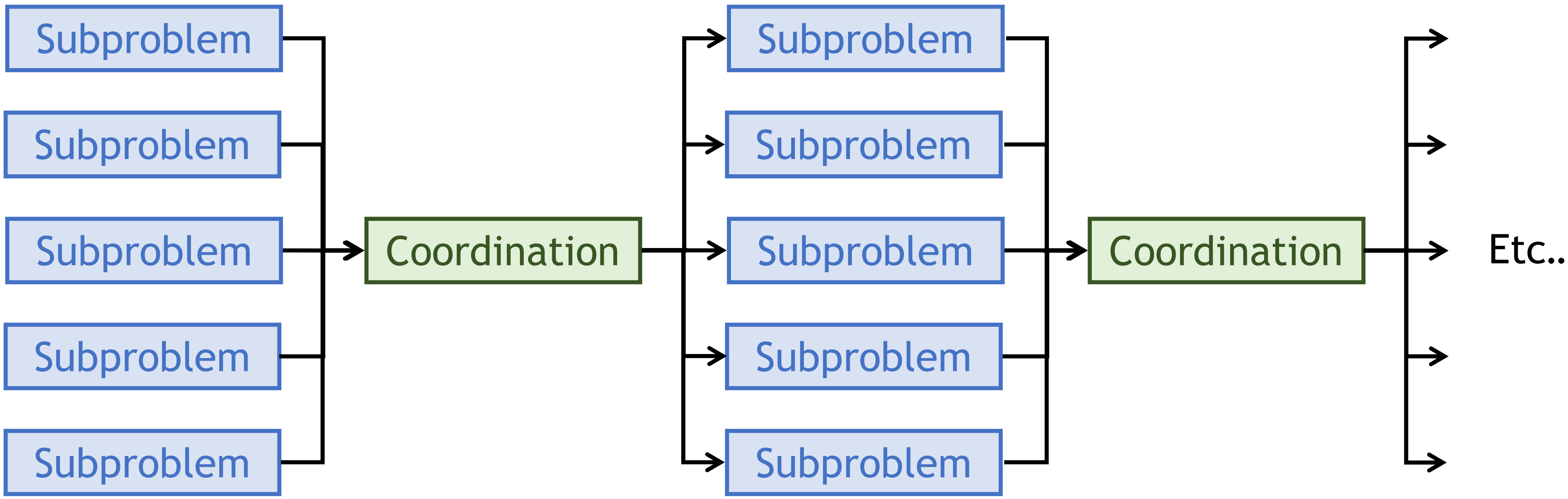
$$\boxed{\begin{array}{ll}
x_i^{k+1} = \underset{x_i \in \mathcal{X}_i}{\arg\min} \left\{ h_i(x_i) + \sum_{s=1}^{T-1} \left(\left\langle x_{is}, w_{is}^k \right\rangle + \frac{\rho}{2} \left\| x_{is} - z_{is}^k \right\|^2 \right) \right\} & i = 1,\ldots,n \\
z_{is}^{k+1} = \dfrac{1}{\left(\sum_{j \in S(i,s)} \pi_j \right)} \sum_{j \in S(i,s)} \pi_j x_{js}^{k+1} & \begin{array}{l} i = 1,\ldots,n \\ s = 1,\ldots,S-1 \end{array} \\
w_{is}^{k+1} = w_{is}^k + \rho(x_{is}^{k+1} - z_{is}^{k+1}) & \begin{array}{l} i = 1,\ldots,n \\ s = 1,\ldots,S-1 \end{array}
\end{array}}$$

- The $w^k \in \mathcal{N}^\perp$ condition can be written as $\sum_{j \in S(i,s)} \pi_j w_{js}^k = 0$ for all i and s

- By using canonical, non-probability-weighted inner products, we can also obtain an alternative version in which simple averages replace the weighted averages and the π_i appear in the x_i minimizations instead

Decomposition Methods

- PH is a form of *decomposition method*
- General form of decomposition methods:



- In any decomposition method, the subproblem computations can be operated in parallel
- But the coordination steps potentially pose a serial bottleneck

Noteworthy Properties of PH

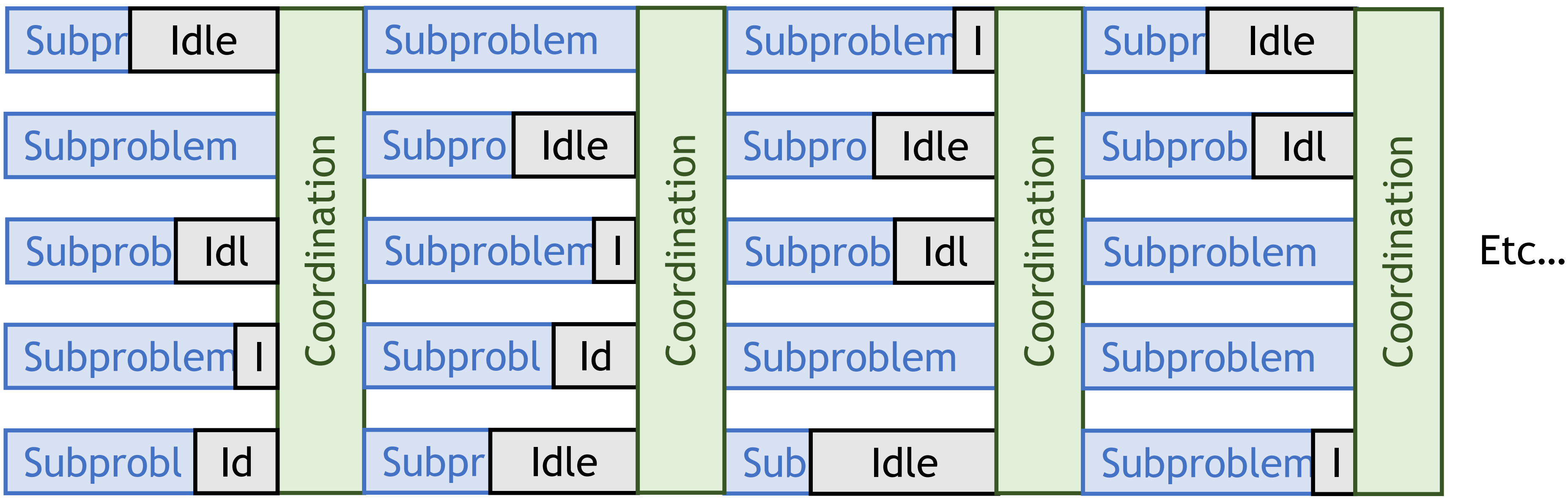
- The coordination computations in PH just consist of sums / averages and simple vector operations
  - These are faster than the "master" optimization problems other decomposition methods typically use…
  - … and can easily be implemented in a distributed manner (efficient parallel algorithms for sums etc.)
- PH handles multi-stage problems cleanly
  - Applying other decomposition methods to problems with 3 or more stages can require unwieldy "nested" versions
- The theory does not require linearity, only convexity
- Superficially, the algorithm is easily adapted to integer variables and nonconvex objectives or constraints
  - Although you lose the standard convergence theory and the method can become heuristic

Adoption of PH in Practice

- Progressive hedging did not “catch on” initially
- Convergence speed on practical problems was not spectacular
- However, its relative simplicity made it start to gain adherents with the advent of
  - Ever-larger problem isntances
  - Interest in problems with many stages
  - Wider availability of highly parallel computing
- So, 20+ years after initial publication, PH started getting used in practice
- The PySP system (Watson, Woodruff, Hart 2018) provides an accessible version of PH coupled with a flexible modeling environment (Pyomo – embedded in Python)
- There has been recent work on making its application with integer variables more rigorous

But Classic PH is Totally Synchronous

- In theory, you must solve every subproblem at every iteration
- The coordination step must wait for the slowest subproblem



Some possible remedies:

- Pack subproblems in processors and load balance (limits parallelism)
- Advanced bundle method variants
- Use projective splitting instead of ADMM / DR (this talk)

Projective Splitting: General Problem Setting

$$0 \in \sum_{i=1}^{n} G_i^* T_i(G_i x)$$

where

- $\mathcal{H}_0, \ldots, \mathcal{H}_n$ are real Hilbert spaces
- $T_i : \mathcal{H}_i \rightrightarrows \mathcal{H}_i$ are maximal monotone operators, $i = 1, \ldots, n$
- $G_i : \mathcal{H}_0 \to \mathcal{H}_i$ are bounded linear maps, $i = 1, \ldots, n$

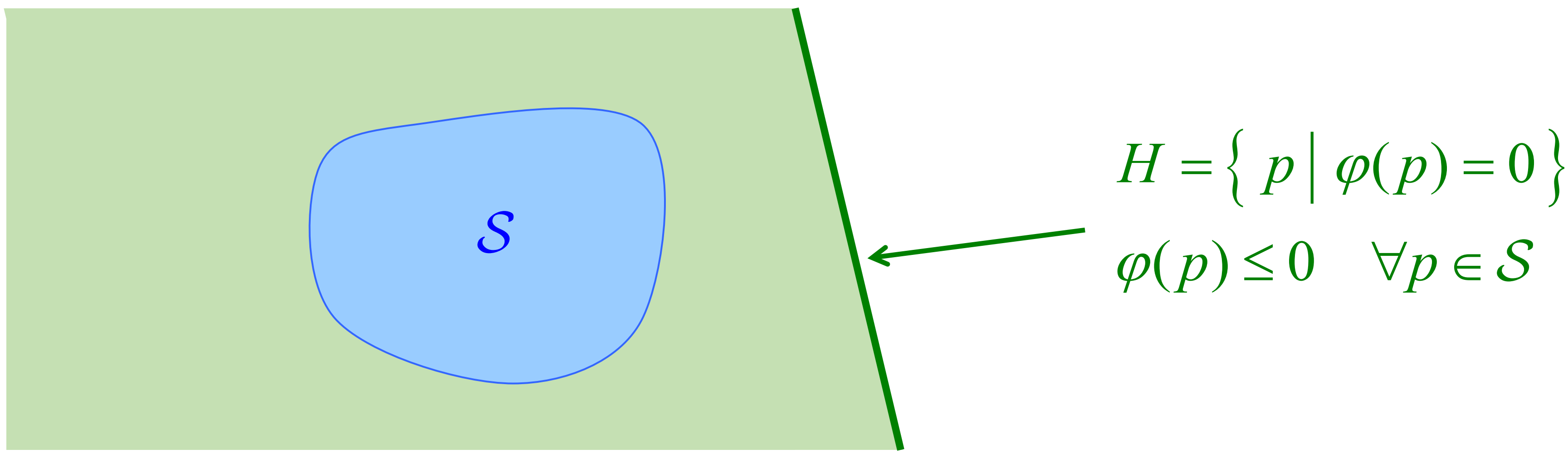
Kuhn-Tucker set / primal-dual solution set

$$\mathcal{S} = \left\{ (z, w_1, \ldots, w_n) \,\middle|\, (\forall i = 1, \ldots n)\; w_i \in T_i(G_i z),\; \sum_{i=1}^{n} G_i^* w_i = 0 \right\}$$

- This is a closed convex set (not immediate; various proofs)

Valid Inequalities for $\mathcal{S}$

- Take some $x_i, y_i \in \mathcal{H}_i$ such that $y_i \in T_i(x_i)$ for $i = 1, \ldots, n$, that is, $(x_i, y_i) \in \operatorname{graph} T_i$
- If $(z, \boldsymbol{w}) = (z, w_1, \ldots, w_n) \in \mathcal{S}$, then $w_i \in T_i(G_i z)$ for $i = 1, \ldots, n$
- So, $\left\langle x_i - G_i z, y_i - w_i \right\rangle \geq 0$ for $i = 1, \ldots, n$ by monotonicity of T_i
- Negate and add up: $\varphi(z, \boldsymbol{w}) = \sum_{i=1}^{n} \left\langle G_i z - x_i, y_i - w_i \right\rangle \leq 0 \quad \forall (z, \boldsymbol{w}) \in \mathcal{S}$



Making Sure these Inequalities are Affine

- Superficially, these inequalities are quadratic
- But with a little care we can make them affine
- One of several possible techniques:
  - Restrict the space to

$$\mathcal{V} = \mathcal{H}_0 \times \mathcal{W} \supset \mathcal{S}, \text{ where } \mathcal{W} = \left\{ \boldsymbol{w} = (w_1, \ldots, w_n) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \;\middle|\; \sum_{i=1}^{n} G_i^* w_i = 0 \right\}$$

  - Within this subspace, φ is affine since the quadratic terms are

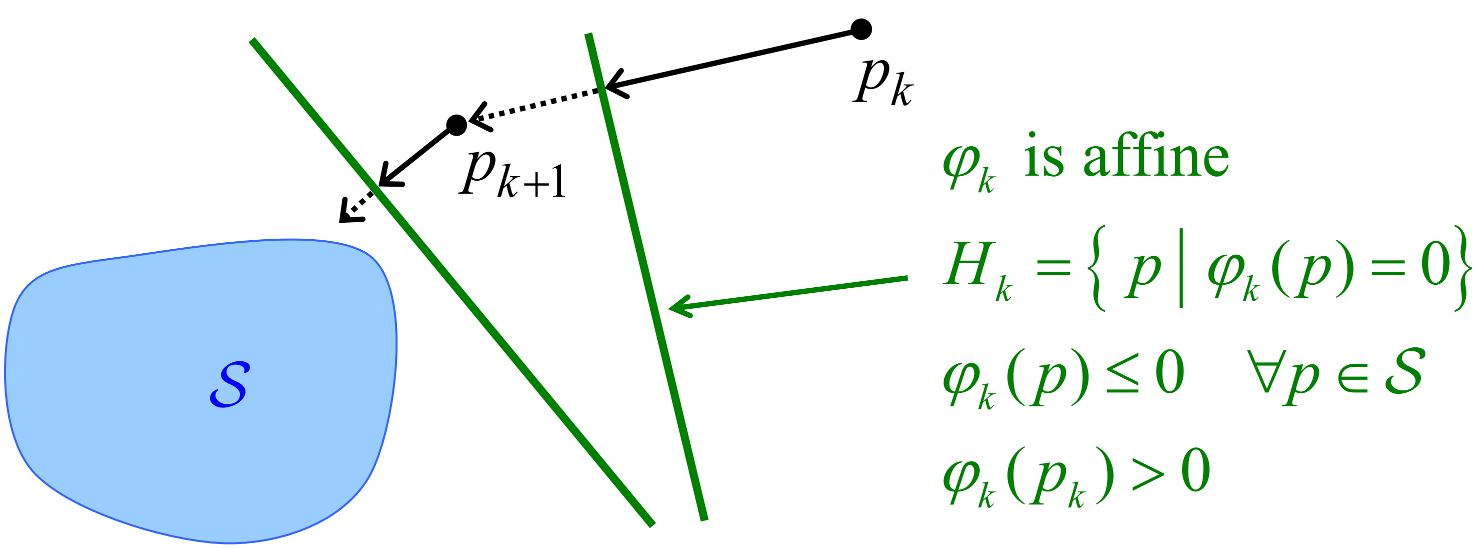
$$\sum_{i=1}^{n} \langle G_i z, -w_i \rangle = \sum_{i=1}^{n} \langle z, -G_i^* w_i \rangle = \left\langle z, -\sum_{i=1}^{n} G_i^* w_i \right\rangle = \langle z, -0 \rangle = 0$$

- Once we know φ is affine, projection onto the halfspace $H = \{ p \in V \mid \varphi(p) \le 0 \}$ is fairly straightforward

Generic Projection Method to Converge to a Point in a Closed Convex Set $\mathcal{S}$ in any Hilbert Space $\mathcal{V}$

Apply the following general template:

- Given $p^k \in \mathcal{V}$, choose some affine function φ_k with $\varphi_k(p) \le 0 \; \forall p \in \mathcal{S}$
- Project p^k onto $H_k = \left\{ p \,\middle|\, \varphi_k(p) \le 0 \right\}$, possibly with an overrelaxation factor $\nu_k \in [\varepsilon, 2-\varepsilon]$, yielding p_{k+1}, and repeat…



Projection Process in the Case of Projective Splitting

- Here, $p^k = (z^k, \boldsymbol{w}^k) = (z^k, w_1^k, \ldots, w_n^k)$ and we find φ_k by picking some $(x_i^k, y_i^k) \in \operatorname{graph} T_i \ (\forall i)$ and using the construction above

$$u^k = \operatorname{proj}_{\mathcal{W}}(x_1^k, \ldots, x_n^k) \qquad v^k = \sum_{i=1}^{n} G_i^* y_i^k$$

$$\tau_k = \left\|u^k\right\|^2 + \gamma \left\|v^k\right\|^2 \quad \text{(done if } \tau_k \approx 0\text{)}$$

$$\theta_k = \frac{\nu_k}{\tau_k} \max\left\{0, \sum_{i=1}^{n} \left\langle G_i z^k - x_i^k, y_i^k - w_i^k \right\rangle \right\}$$

$$z^{k+1} = z^k - \frac{\theta_k}{\gamma} v^k \qquad w^{k+1} = w^k - \theta_k u^k$$

- There are alternative approaches if $\operatorname{proj}_{\mathcal{W}}$ is difficult
- $\gamma > 0$ is an optional primal-dual scaling factor
- More complicated than ADMM/PH coordination step, but still just simple vector and sum operations (so could be distributed)

How to Pick the x_i^k, y_i^k — Basics

- If you pick $(x_i^k, y_i^k) \in \operatorname{graph} T_i$ completely arbitrarily, you may just orbit around $\mathcal{S}$ and not converge to it
- A workable choice: the "prox" operation for some scalar $c_{ik} > 0$

$$\boxed{(x_i^k, y_i^k) = \operatorname{Prox}_{T_i}^{c_{ik}}(G_i z^k + c_{ik} w_i^k)}$$

That is,

$$x_i^k = (I + c_{ik} T_i)^{-1}(G_i z^k + c_{ik} w_i^k) \qquad y_i^k = \frac{1}{c_{ik}}\left(G_i z^k + c_{ik} w_i^k - x_i\right)$$

- Then $c_{ik}(y_i^k - w_i^k) = G_i z^k - x_i^k$
- So $\left\langle G_i z^k - x_i^k, y_i^k - w_i^k \right\rangle = c_{ik}\left\| G_i z^k - x_i^k \right\|^2 = c_{ik}^{-1}\left\| y_i^k - w_i^k \right\|^2 \geq 0$
- Sum over i and get $\varphi_k(z^k, \boldsymbol{w}^k) > 0$ (cuts of current iterate)
- Can prove that this guarantees (weak) convergence to $\mathcal{S}$ if the c_{ik} are bounded away from zero and infinity

How to Pick the x_i^k, y_i^k — "Block Iterativity"

- Variation: do not have to activate every operator at every iteration (Combettes & E 2018)
  - For the rest, just recycle the previous x_i^k, y_i^k
- Let $M \geq 0$ be an integer
- Let $I_0, I_1, I_2, \ldots \subseteq \{1, \ldots, n\}$ be such that

$$(\forall i \geq 0) \quad \bigcup_{j=i}^{i+M} I_j = \{1, \ldots, n\}$$

- At iteration k, only activate the operators in I_k :

$$(x_i^k, y_i^k) = \mathrm{Prox}_{T_i}^{c_{ik}}(G_i z^k + c_{ik} w_i^k) \quad \forall i \in I_k$$
$$(x_i^k, y_i^k) = (x_i^{k-1}, y_i^{k-1}) \quad \forall i \in \{1, \ldots n\} \setminus I_k$$

- Convergence proof adapts ideas from successive projection methods for set intersection problems

How to Pick the x_i^k, y_i^k – "Lags"

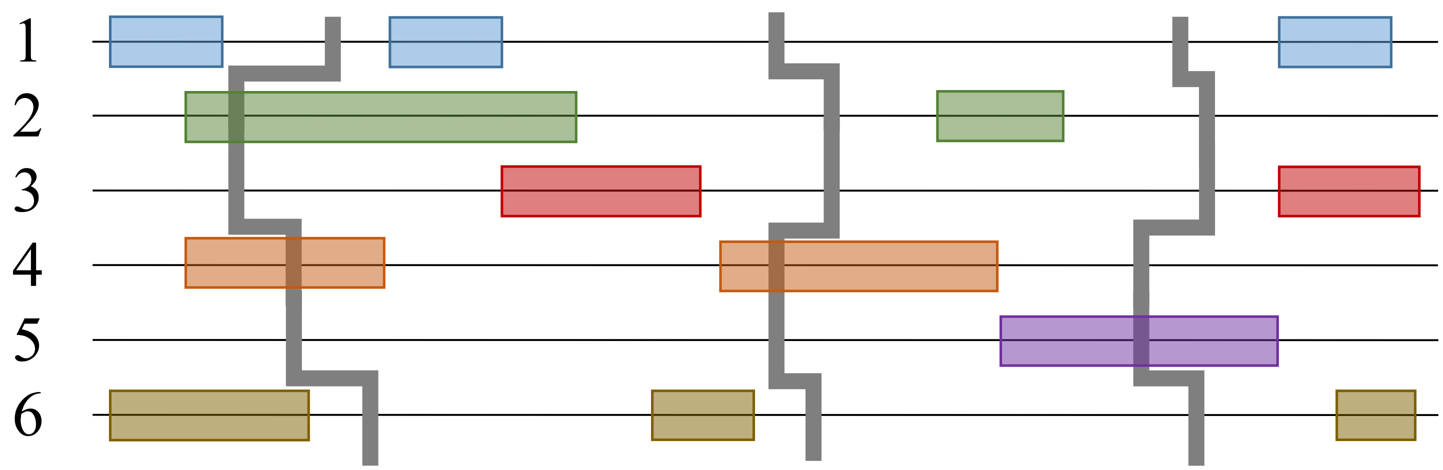
- Also from Combettes & E (2018)
- Let $K \geq 0$ be another integer
- Each prox operation may use data from up to K iterations ago
- Otherwise same as above

- So, for some $d(i,k)$ with $k \geq d(i,k) \geq k - K$,

$$
\begin{aligned}
(x_i^k, y_i^k) &= \mathrm{Prox}_{T_i}^{c_{ik}}\left(G_i z^{d(i,k)} + c_{ik} w_i^{d(i,k)}\right) & \forall i \in I_k \\
(x_i^k, y_i^k) &= (x_i^{k-1}, y_i^{k-1}) & \forall i \in \{1,\ldots n\} \setminus I_k
\end{aligned}
$$

Asynchrony (Sloppy Notation)

- Combining block iterativity and lags lets one drop strict coupling of coordination and subproblem processing
- For each i, suppose that a new $(x_i, y_i) \leftarrow \mathrm{Prox}_{T_i}^{c}(G_i z + c w_i)$ appears at least every t_1 time units, based on (z, w_i) that are at most t_2 time units old
- A projection step occurs at least every t_3 time units, based on $(x_i, y_i) \in \mathrm{graph}\, T_i,\ i = 1, \ldots, n$ that are most t_4 time units old
- Then $(z, \boldsymbol{w})$ converges (weakly) to a point in $\mathcal{S}$



Applying Projective Splitting to Stochastic Programming

Problem setup for stochastic programming

- $\mathcal{H}_0 = \mathcal{N}$ (run algorithm in nonanticipativity subspace)
- $\mathcal{H}_i = \mathcal{Z}_i$, but with its inner product multiplied by π_i
- $G_i : \mathcal{N} \to \mathcal{Z}_i$ selects the subvector relevant to scenario i
- $f_i(\tilde{x}_i) = \min\limits_{x_{iT}} \left\{ \pi_i h_i \left((\tilde{x}_i, x_{iT}) \right) \right\}$ minimizes scenario i's cost over the last-stage variables
    - Remember, scenario-infeasible points have $h_i(x_i) = +\infty$

Then our stochastic program is just

$$\min_{x \in \mathcal{H}_0} \sum_{i=1}^{n} f_i(G_i x)$$

So apply the method from earlier in the talk for $0 \in \sum_{i=1}^{n} G_i^* \partial f_i(G_i x)$

Some Technicalities

- This choice of f_i is convex for convex h_i
- But it is not generally guaranteed to be closed unless h_i has compact effective domain
- We need such an assumption to guarantee that f_i is closed and thus that $T_i = \partial f_i$ is maximal

- We also need some constraint-qualification-like conditions to be sure that the sufficient condition $0 \in \sum_{i=1}^{n} G_i^* \partial f_i(G_i x)$ is also necessary for optimality
  - Should not be a major concern in practice

Subproblem Processing

Subproblem: (many operating in parallel, asynchronously)

Let $0 < \rho_{\min} \le \rho_{\max} < \infty$ be fixed

Parameters for subproblem i:

- $z_i = (z_{i1}, \ldots, z_{i,T-1})$: scenario i "target" values, except last stage
- w_i : multipliers (same dimensions as z_i)

Get recent $z_i, w_i \in \mathcal{Z}_i$ from coordination process,

Select some $\rho \in [\rho_{\min}, \rho_{\max}]$

Let $x_i \in \operatorname*{Arg\,min}_{x_i} \left\{ h_i(x_i) + \langle M_i x_i, z_i \rangle + \frac{\rho}{2} \| M_i x_i - z_i \|^2 \right\}$

and $y_i = w_i + \rho(M_i x_i - z_i)$

Make i, $\tilde{x}_i \doteq M_i x_i$, y_i available to coordination process

Looks like PH subproblem + part of multiplier update

Coordination Process Variables

The coordination process maintains working variables:

- $z = (z_1, \ldots, z_n) \in \mathcal{N}$
- $w = (w_1, \ldots, w_n) \in \mathcal{N}^\perp$
- $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathcal{Z}$ (the tildes mean no last-stage variables)
- $y = (y_1, \ldots, y_n) \in \mathcal{Z}$

At each iteration we also compute step direction vectors:

- $u = (u_1, \ldots, u_n) \in \mathcal{N}^\perp$
- $v = (v_1, \ldots, v_n) \in \mathcal{N}$

Scalar parameters:

- Primal-dual scaling factor $\gamma > 0$ (fixed?)
- Overrelaxation factor limits $0 < \nu_{\min} \le \nu_{\max} < 2$ (varying)

Coordination Process

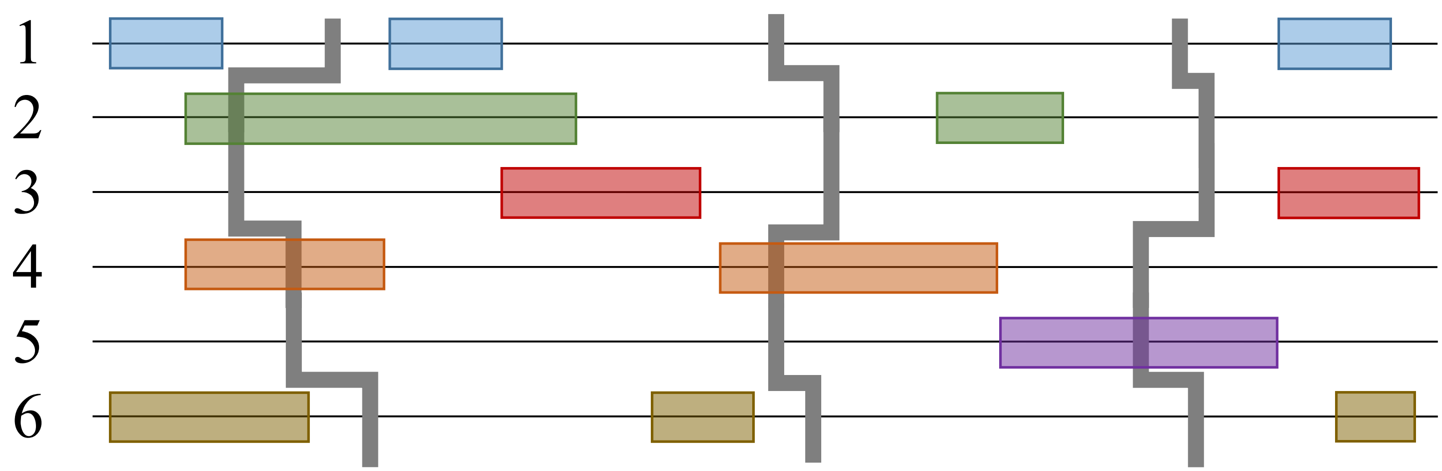
repeat
for $i = 1, \ldots, n$
let $\tilde{x}_i, y_i \in \mathcal{Z}_i$ be recent values from subproblem i
and let $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ and $y = (y_1, \ldots, y_n)$
$u \leftarrow \tilde{x} - \mathrm{proj}_{\mathcal{N}}(\tilde{x})$
$v \leftarrow \mathrm{proj}_{\mathcal{N}}(y)$
$\tau \leftarrow \|u\|^2 + \gamma \|v\|^2 = \sum_{i=1}^{n} \pi_i \|u_i\|^2 + \gamma \sum_{i=1}^{n} \pi_i \|v_i\|^2$
$\phi \leftarrow \langle z - \tilde{x}, w - y \rangle = \sum_{i=1}^{n} \pi_i \left(z_i - \tilde{x}_i\right)^{\mathsf{T}} \left(w_i - y_i\right)$
if $\phi > 0$ **then**
Choose some $\nu \in [\nu_{\min}, \nu_{\max}]$
$z \leftarrow z + (\nu\phi / \tau\gamma) v$
$w \leftarrow w + (\nu\phi / \tau) u$
until termination detected

- **Note:** this is not necessarily a central "master" process; it can be distributed

Asynchrony

Same conditions for convergence as in abstract asynchronous case above:

- Each subproblem is recomputed least every t_1 time units, based on (z, w_i) that are at most t_2 time units old
- A coordination step completes at least once every t_3 time units, based subproblem results that are at most t_4 time units old
- Then $(z, \mathbf{w})$ converges to a primal-dual solution, as in PH



Asynchronous Projective Hedging

- We call the resulting class of algorithms *asynchronous projective hedging* (APH)

Partial Resemblance to PH

- Subproblem has some recognizable pieces of the PH subproblem optimization step and multiplier update
  - Essentially the same minimization step for subproblems
  - $\mathrm{proj}_{\mathcal{N}}$ and simple vector operations
- The control process is somewhat more complicated than PH, but consists of the same fundamental operations
  - Nothing more complicated than $\mathrm{proj}_{\mathcal{N}}$
  - May still be implemented in a distributed way

But Now It's Asynchronous

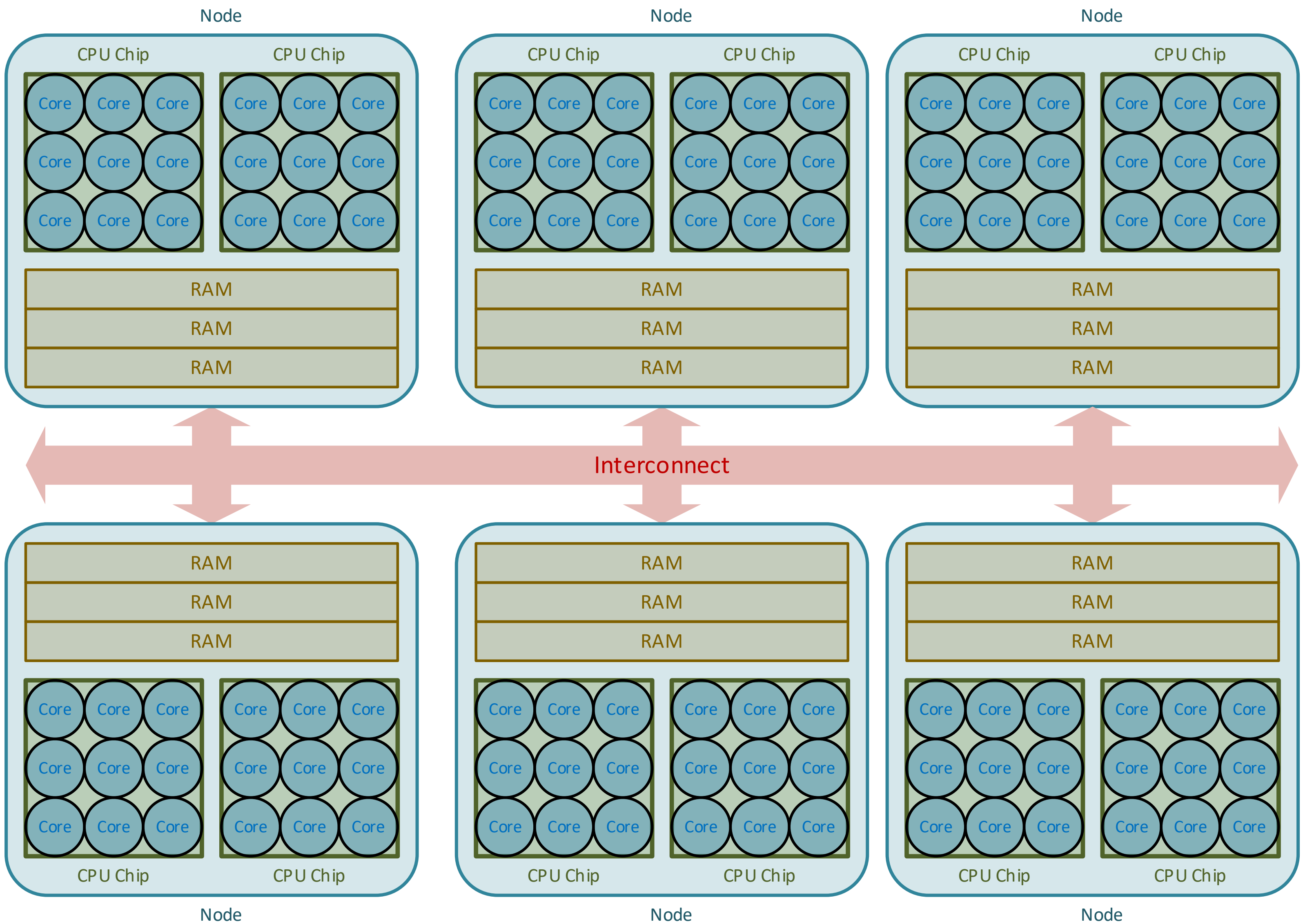
- Full synchronization has been replaced by loose timing bounds
- The subproblem and coordination processes can run at different speeds
  - No longer "locked in" to one coordination for every n subproblem solves
  - Now possible to have

$$\frac{\text{\# subproblem solves initiated by time } t}{\text{\# coordination steps by time } t} < n$$

Application

- **Problem:** SSN telecommunication design problem (Sen et al. 1994)
  - Standard test problem class in stochastic programming
  - For this exercise, we generated instances with up to 10^6 sample scenarios
  - Underlying number of scenarios finite but $\approx 10^{70}$
- **Hardware:** "Quartz" supercomputer at Lawrence Livermore
- **Software platform:** mpi-sppy (Kneuven et al. 2020)

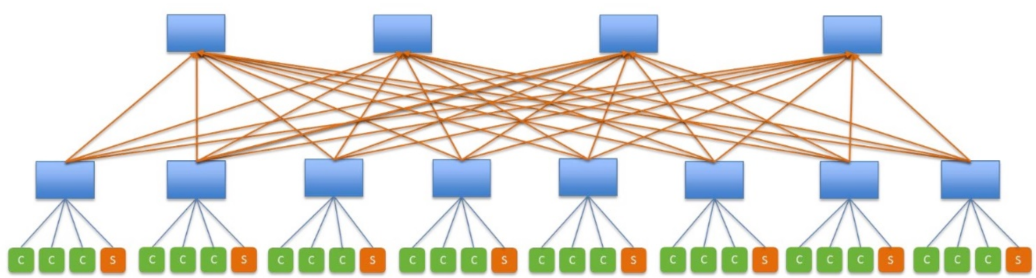
Standard Modern Supercomputer



- CPU cores share memory within each node
- Nodes communicate by messages through an interconnect

Quartz

- 32 CPU cores/node
- 128 GB RAM/node
- About 3,000 nodes
- Omni-Path interconnect (channel speed 100 G bits/second)



- Our biggest job so far used 250 nodes / 8,000 cores

mpi-sppy

- Sandia/Livermore package for stochastic programming
- Built on Pyomo optimization modeling environment that embeds in Python
- Coded in Python, but
  - Most CPU time spent solving subproblems (Gurobi etc.)
  - Or within numpy linear algebra kernels (calling BLAS)
- Has a “hub and spoke” architecture
  - But not in the classic “master-slave” sense
- “MPI” is how messages get sent between groups of processors (Gropp et al. 2014)

Lower Bounds

- Neither PH nor APH immediately provide pre-termination lower bounds on the optimal solutions value
- PH never fully minimizes the augmented Lagrangian, so it does not automatically provide a Lagrangian bound
- APH is similar
- But one can obtain one by doing an extra minimization of the ordinary Lagrangian (separable) – since $w^k \in \mathcal{N}^\perp$, compute

$$L(w^k) = \sum_{i=1}^{n} \min_{x_i} \left\{ f_i(x_i) + \left\langle M_i x_i, w_i^k \right\rangle \right\}$$

- And there may be other, application-specific lower bounds
- There is also a bound (E 2020) that one can derive directly from the PH process, but it requires estimation of a potentially large constant and may not be readily applicable

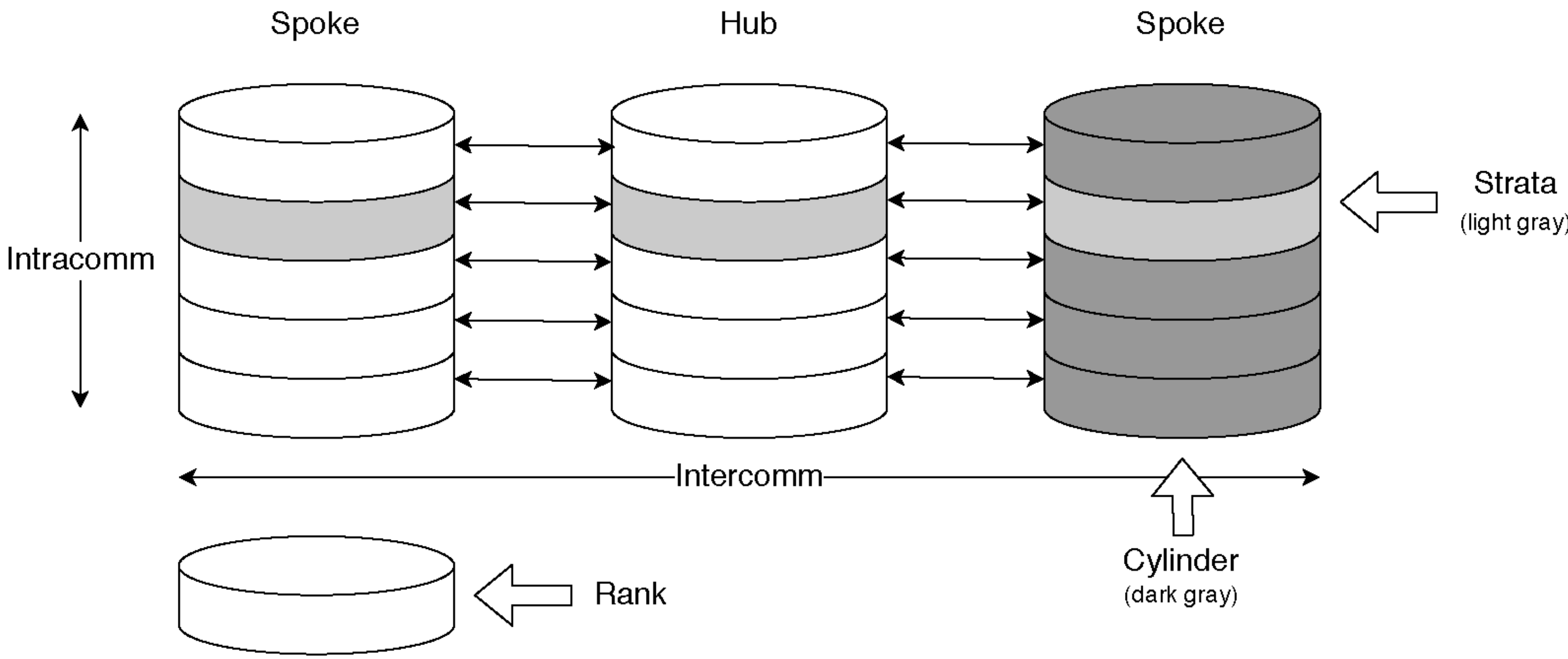
Upper Bounds

- Neither PH nor APH provide pre-termination feasible solutions
- Nonanticipativity is only satisfied in the limit
- Various strategies for deriving these feasible solutions from z

mpi-sppy Processor Organization

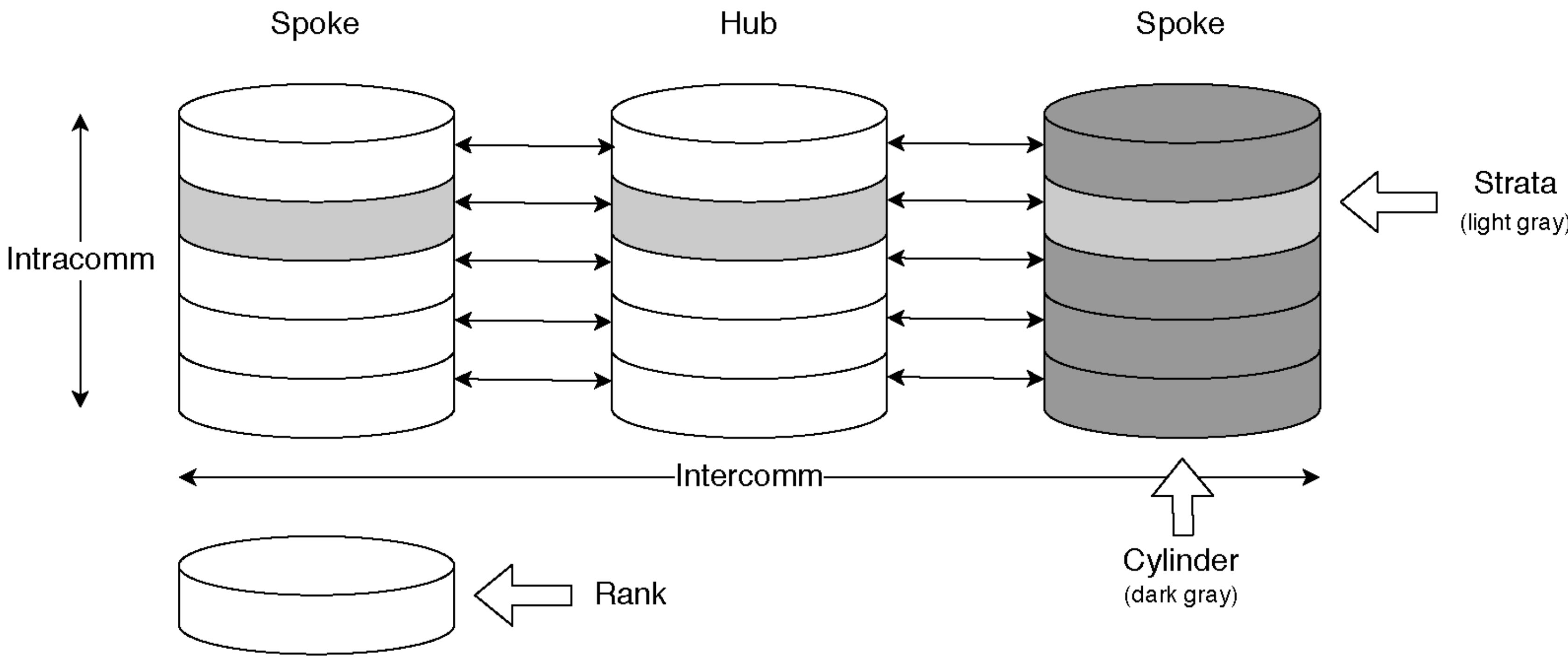
- A “hub” grouping of processors (possibly very large) runs the principal optimization algorithm (PH or APH)
- “Spoke” groupings of processors run auxiliary processes like upper and lower bounding (possibly several of each)
- The spoke operations are seeded from the principle $(z, \boldsymbol{w})$ iterates from the hub
- The spokes do not need to communicate directly with one another

The mpi-sppy Picture



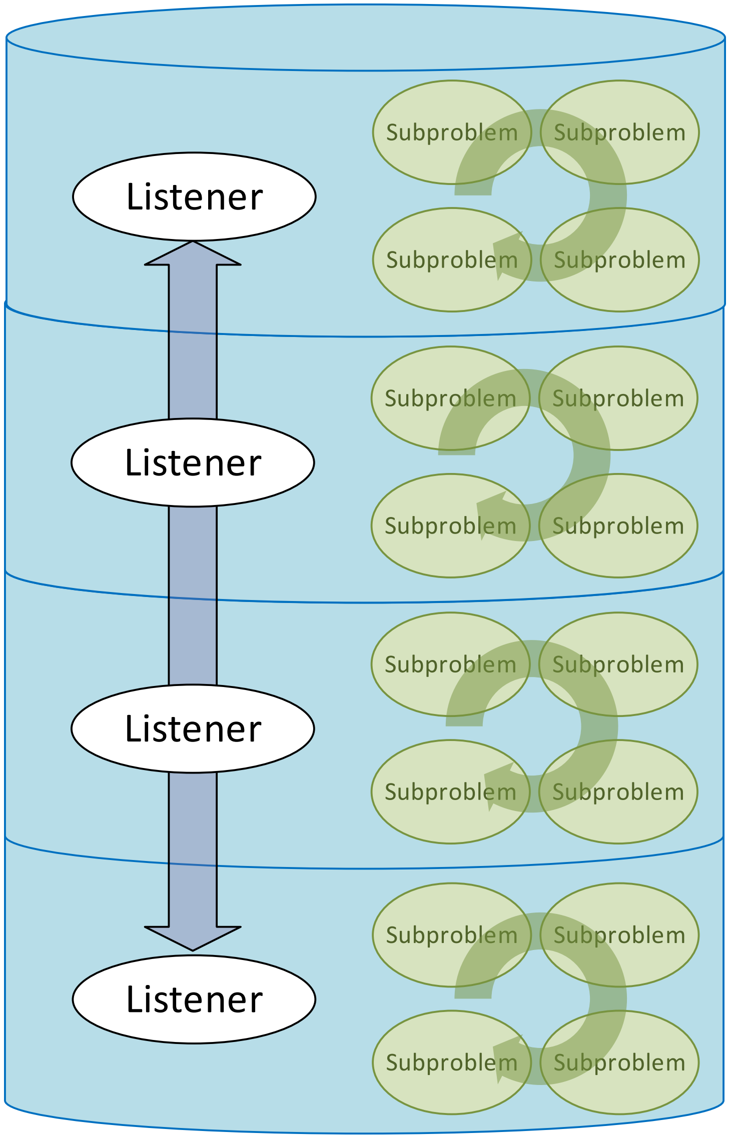
- A *rank* is a single shared memory space
  - Typically ~4 CPU cores (the most Gurobi efficiently can use for QPs)
  - Pack multiple ranks onto a node
  - Runs multiple threads and typically has multiple CPU cores
- Ranks are organized into *cylinders* and *strata*

The mpi-sppy Picture



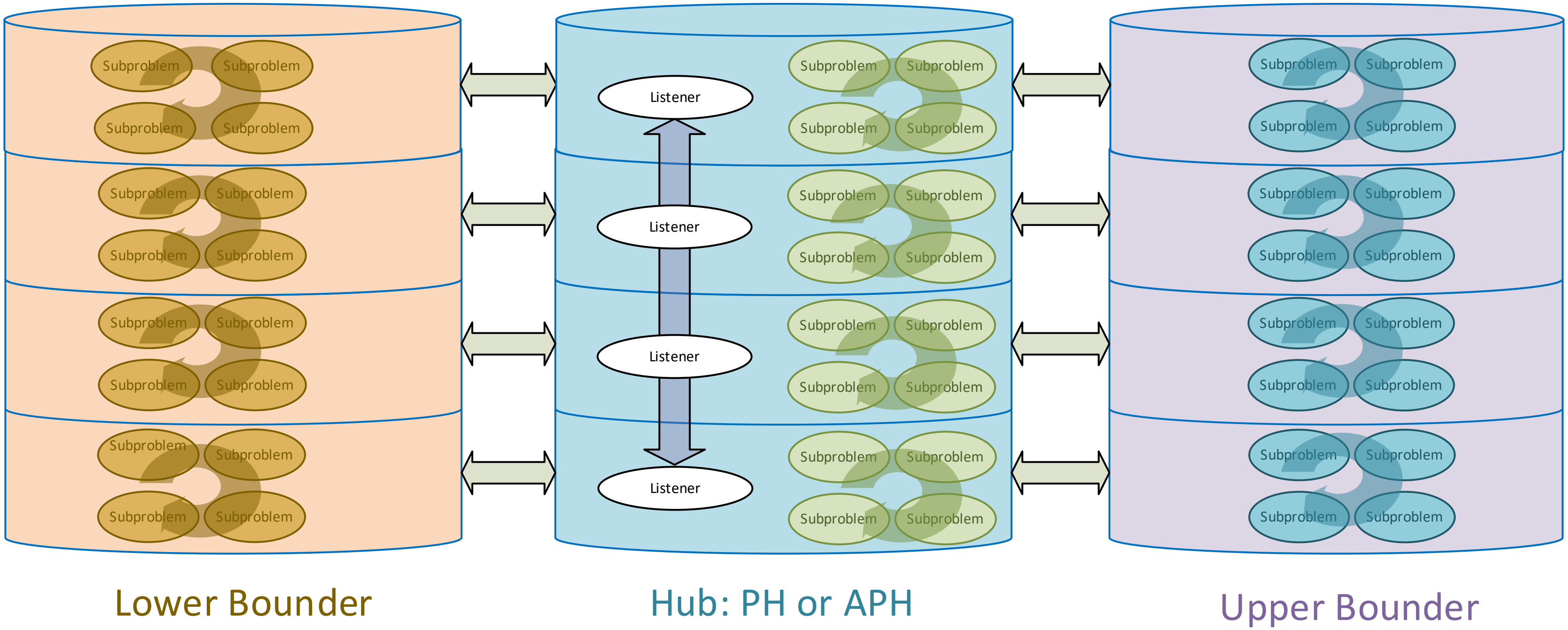
- Each rank stores the data for one or more scenarios
- Within a cylinder, each scenario is stored on only one rank
- Each cylinder has the same number of ranks
- The corresponding ranks in each cylinder (a stratum) each store data for the same scenarios (redundantly)

Within the Hub Cylinder



- Ranks may store more than one scenario
- Each rank only solves one scenario subproblem at a time
  - But Gurobi and numpy may employ multiple cores when doing so
- In PH, synchronous communication once every scenario has been solved
  - Subproblems stop during the communication
- In APH, an “listener” thread decides when “enough” scenarios have been solved (globally), then performs the communication needed for coordination
  - All in parallel with subproblem solves

Hub and Spoke



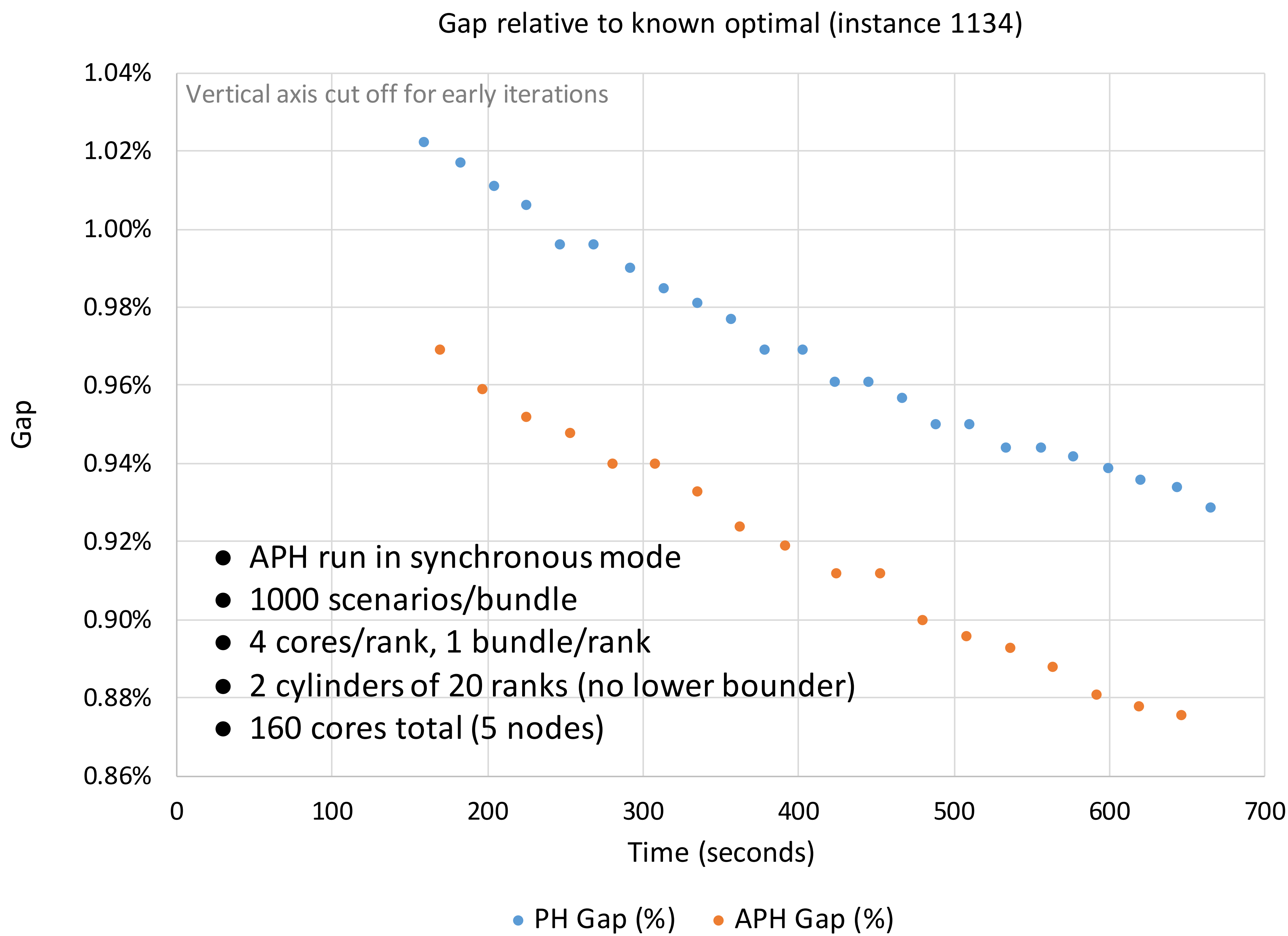
- The spokes are organized similarly to the hub
- Hub periodically sends z and/or w information to spokes
- Each rank sends or receives data only for the subproblems it owns
- Messages in different strata move in parallel

A Few More Points - Bundling and Subproblem Dispatch

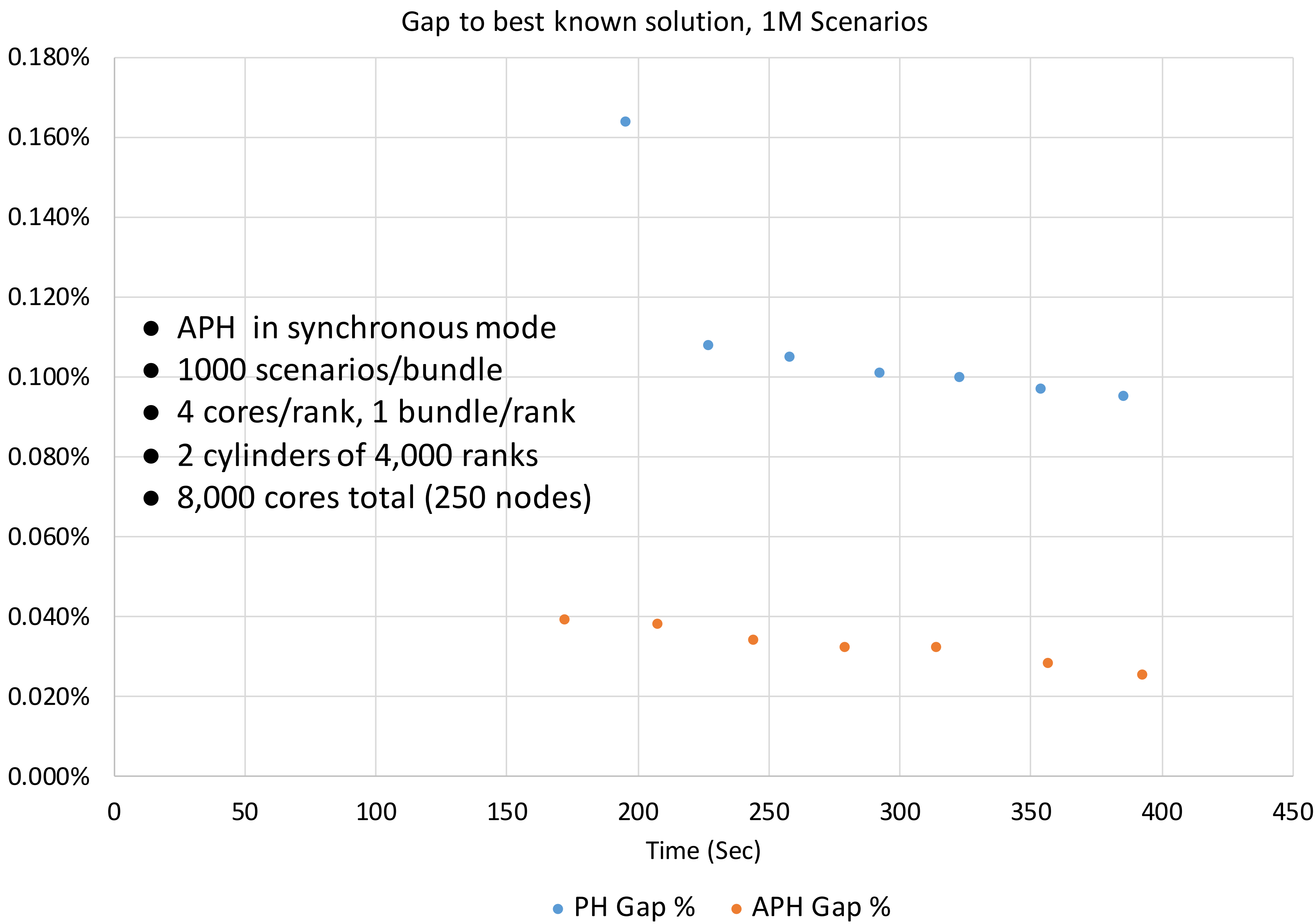
- For large-scale problems, it is common to use a “bundling” strategy
- Single-scenario subproblems are replaced by *bundles* of multiple scenarios
- Within a bundle/subproblem, scenarios are linked by explicit constraints (a “mini extensive form”)
- But the logic of nonanticipativity contraints (now between bundles) and PH / APH remain essentially the same

- In APH, we have a heuristic for selecting the most promising subproblem i to solve next within each rank
  - Based on value of $\pi_i \left(z_i - \tilde{x}_i \right)^{\mathsf{T}} \left(w_i - y_i \right)$

Preliminary Results: 20,000 Scenarios



Preliminary Results: 1,000,000 Scenarios



- APH in synchronous mode
- 1000 scenarios/bundle
- 4 cores/rank, 1 bundle/rank
- 2 cylinders of 4,000 ranks
- 8,000 cores total (250 nodes)

Size of the 1,000,000 Scenario Problem

The equivalent extensive-form LP would have about

- 795 million variables
- 265 million constraints (not counting simple variable bounds)
- 2.64 billion nonzero matrix entries

Much More to Do

- Dual instead of primal derivation of the APH method?
  - Would be more like classic derivation of ADMM from DR
  - Might eliminate annoying assumptions like a compact domain for the h_i
- Investigate making projective scaling methods in general more robust to problem scaling etc.
- The real problems the energy labs want to solve have nonconvexities and integer variables
  - Related to operating electric power grids
  - How can we help address such problems…
    - Rigorously?
    - Heuristically?
  - Asynchrony is an especially nice feature when the subproblems have integer variables