Robust Interior Point Methods and FR for Key Rate Computation in Quantum Key Distribution

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Motivation/Outline

- find reliable, efficient numerical method for calculating key rates for quantum key distribution (QKD) protocols
- Currently: ill-posed models; i.e., we want to (minimize)

  find reliable provable lower bound

  for the convex relative entropy: $\operatorname{trace} \rho \log \rho - \sigma \log \rho$

  $\sigma, \rho \succeq 0$ (positive semidefinite matrices), even though singular (opt. currently on boundary of SDP cone)

regulariz. using FACIAL REDUCTION, FR; on both constraints and nonlinear objective

- (I) theoretically proven upper and lower bounds with possible approximate FR; high precision
- (II) (Gauss-Newton) interior point approach on regularized problem; (originally singularity degree ONE>0)
- avoid current perturbation approach to get $\rho \succ 0$ (pos. def.)

QKD Background (Details in References)

- Quantum key distribution, QKD: the art of distributing secret keys between two honest parties, traditionally known as Alice and Bob;
- secret key rate (number of bits of secret key obtained per exchange of quantum signal) calculation is at the core of a security proof for any QKD protocol;
- calculation is a convex minimization (lower bound) problem, s.t. constraints to detect presence of any third party (Eve eavesdropping); fundamentally: security comes from the Heisenberg uncertainty principle as eavesdropping means detectable disturbances so Alice and Bob can detect presence of Eve;
- even with a quantum computer, a secret key generated by QKD remains secure.

(asymptotic) Key Rate Calculation

Winick, Lütkenhaus, Coles [9]

$$\begin{array}{rll} p^* = & \min_\rho & D(\mathcal{G}(\rho)\|\mathcal{Z}(\mathcal{G}(\rho))) \\ & \text{s.t.} & \Gamma(\rho) = \gamma, \quad (\operatorname{trace}\rho = 1) \\ & & \rho \succeq 0 \qquad \text{(density matrices)} \end{array}$$

Here:

- $D(\delta\|\sigma) = f(\delta,\sigma) = \operatorname{trace}\delta[\log\delta - \log\sigma]$ is the quantum relative entropy;
- $\Gamma : \mathbb{H}^n \to \mathbb{R}^m$ lin. transf., $\quad \Gamma(\rho) = (\operatorname{trace}\Gamma_i\rho) = (\langle\Gamma_i, \rho\rangle)$;
- $\mathbb{H}^n$ linear space Hermitian matrices over $\mathbb{R}$; $\quad \gamma \in \mathbb{R}^m$
- $\mathcal{G}$ and $\mathcal{Z}$ are linear, completely positive maps, CP (here, sums of products $Z_i\rho Z_i^*$)

CP $\mathcal{G}, \mathcal{Z}$; e.g., $\mathcal{G} : \mathbb{H}^n \to \mathbb{H}^k$, $k > n$, $\mathcal{G}(\mathbb{H}^n_+) \subseteq \mathbb{H}^k_+$

e.g. $\mathcal{G}(\rho) = \sum_{j=1}^t K_j\rho K_j^*, \quad \sum_{j=1}^t K_j^* K_j \preceq I$.

Linear Maps $\mathcal{G}, \mathcal{Z}$

Definition ($\mathcal{G} : \mathbb{H}^n \to \mathbb{H}^k$ (Kraus repres.))

$$\mathcal{G}(\rho) := \sum_{j=1}^{\ell} K_j \rho K_j^*,$$

$K_j \in \mathbb{C}^{k \times n}$, $\sum_{j=1}^{\ell} K_j^* K_j \preceq I$; adjoint $\mathcal{G}^*(\delta) := \sum_{j=1}^{\ell} K_j^* \delta K_j$;
Generally $k = i * n > n, i = 2, 3, \ldots;$
and so typically $\mathcal{G}(\rho)$ rank deficient $\forall \rho \succ 0$ (positive definite)

Definition (self-adjoint (projection) $\mathcal{Z} : \mathbb{H}^k \to \mathbb{H}^k$)

$$\mathcal{Z}(\delta) := \sum_{j=1}^{N} Z_j \delta Z_j,$$

$Z_j = Z_j^2 = Z_j^* \in \mathbb{H}_+^k$ and $\sum_{j=1}^{N} Z_j = I_k$

Properties of QKD Problem

Lemma

The linear map $\mathcal{Z}$ *is an orthogonal projection on* $\mathbb{H}^k$ *and* $\text{trace}(\delta) \leq 1, \delta \succ 0$ *implies:*

$$\text{trace}\left(\delta \log \mathcal{Z}(\delta)\right) = \text{trace}\left(\mathcal{Z}(\delta) \log \mathcal{Z}(\delta)\right)$$

$\mathcal{G}$, $\mathcal{Z} \circ \mathcal{G}$ May not Preserve Positive Definiteness

$$\begin{array}{rll} p^* = & \min_\rho & D(\mathcal{G}(\rho) \| \mathcal{Z}(\mathcal{G}(\rho))) \\ & \text{s.t.} & \Gamma(\rho) = \gamma, \\ & & \rho \succeq 0 \qquad \text{(density matrices)} \end{array}$$

Known Properties

$$\min_{\rho,\sigma,\delta} \text{trace}(\delta(\log \delta)) - \text{trace}(\sigma(\log \sigma))$$

quantum relative entropy D is finite under range condition; jointly convex in both δ and σ

Equivalent Formulation

Ready for FR, Facial Reduction (Regularization)

$$
\begin{array}{rll}
p^* = & \min_{\rho,\sigma,\delta} & \operatorname{trace}(\delta(\log \delta)) - \operatorname{trace}(\sigma(\log \sigma)) \\
& \text{s.t.} & \Gamma(\rho) = \gamma \\
& & \sigma = \mathcal{Z}(\delta) \\
& & \delta = \mathcal{G}(\rho) \\
& & \rho, \sigma, \delta \succeq 0.
\end{array}
$$

Our Goal: Final Asymptotic Key Rate

obtained by getting a reliable lower bound of this problem (and then removing the cost of error correction, a constant).

Facial Reduction, FR, Borwein-W. [3], Preliminaries

Slater Constraint Qualification, Strict Feasibility, Stability

Slater: $\exists \hat{\rho} \succ 0 : \Gamma \hat{\rho} = \gamma$

Strong Duality, Stability

Slater is sufficient for strong duality;
equivalent to numerical stability under RHS perturbations;
Slater fails in surprisingly many applications

Advantages of FR

FR can be used to obtain strict feasibility and regularize the problem, and often simultaneously simplify the problem.

current applications: e.g.

hard discrete opt.; (distance geometry); EDM and low rank matrix completion etc ... (recent survey Drusvyatskiy-W. [5])

Facial Reduction, FR, Preliminaries cont...

convex cone: $K : \lambda K \subseteq K, \forall \lambda \geq 0,\ K + K \subset K$,
dual cone: $S^* = \{\phi \in \mathbb{H} : \langle \phi, s \rangle \geq 0,\ \forall s \in S\}$.
convex cone F is a face of a convex cone K, $F \trianglelefteq K$, if

$$x, y \in K, x + y \in F \implies x, y \in F.$$

Faces of the positive semidefinite cone are characterized by the range or nullspace of any element in the relative interior:

Lemma

Let F a convex subset of $\mathbb{H}^n_+$ with $X \in \operatorname{ri} F$ with orthogonal spectral decomposition $X = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & Q \end{bmatrix}^*$, with $D \in \mathbb{H}^r_{++}$. Then TFAE:

(i) $F \trianglelefteq \mathbb{H}^n_+$;

(ii) $F = \{Y \in \mathbb{H}^n_+ : \operatorname{range}(Y) \subset \operatorname{range}(X)\}$
$\quad = \{Y \in \mathbb{H}^n_+ : \operatorname{null}(Y) \supset \operatorname{null}(X)\}$;

(iii) $F = P\mathbb{H}^r_+ P^*$; *(iv)* $F = \mathbb{H}^n_+ \cap (QQ^*)^\perp$ *(exposing QQ^*)*

Facial Reduction via Theorem of Alternative

Lemma (theorem of the alternative)

For the feasible constraint system, exactly one of the following statements holds:

1. *there exists* $\rho \succ 0$ *such that* $\Gamma(\rho) = \gamma$ *(Slater);*
2. *there exists* y *(and exposing vector* Z*) such that*

$$0 \neq Z = \Gamma^*(y) \succeq 0 \ , \ \langle \gamma, y \rangle = 0.$$

The matrix $Z = \Gamma^* y$ above is an exposing vector for the feasible set.

Definition (minimal face)

K a closed convex cone; $S \subseteq K$ a convex set; then $\mathrm{face}(S) \unlhd K$ is the *minimal face*, the intersection of all faces of K that contain S.

First (Partial) FR Step on S_R (Analytical/Accurate)

affine manifold constraint is divided into two sets

observable, reduced density operator constraint sets, $S_O \cap S_R$; with Kronecker product, $\otimes$

$$S_O = \left\{ \rho \succeq 0 \,:\, \langle P_s^A \otimes P_t^B, \rho \rangle = p_{st}, \, \forall s, t \right\},$$

$$\begin{array}{rcll} S_R & = & \{ \rho \succeq 0 \,:\, \mathrm{trace}_B(\rho) = \rho_A \} & \text{(partial trace)} \\ & = & \left\{ \rho \succeq 0 \,:\, \langle \Theta_j \otimes \mathrm{I}_B, \rho \rangle = \theta_j, \, \forall j = 1, \ldots, m_R \right\}, & \end{array}$$

where

data $\theta_j = \langle \Theta_j, \rho_A \rangle$; and $\rho_A \in \mathbb{H}_+^{n_A}$ often singular
$\{\Theta_j\}$ orthonormal basis system A.

Let range $P = \mathrm{range}\, \rho_A \subsetneq \mathbb{H}^{n_A}$, $P^*P = \mathrm{I}_r$, and let $V = P \otimes \mathrm{I}_B$.
Then $\boxed{\mathrm{FR}: \; \rho \in S_R \implies \rho = VRV^*, \text{ for some } R \in \mathbb{H}_+^{r \cdot n_B}}$

FR on the Objective Function

Lemma (useful equivalent form for entropy function)

Let $Y = VRV^* \in \mathbb{H}_+$, $R \succ 0$ *be the compact spectral decomposition of* Y *with* $V^*V = I$. *Then*

$$\operatorname{trace}(Y \log Y) = \operatorname{trace}(R \log R).$$

Proof.

We obtain a unitary matrix $U = \begin{bmatrix} V & P \end{bmatrix}$ by completing the basis. Then $Y = UDU^*$, where $D = \operatorname{BlkDiag}(R, 0)$. We conclude, with $0 \cdot \log 0 = 0$, that

$\operatorname{trace} Y \log Y = \operatorname{trace} D \log D = \operatorname{trace} R \log R$. □

Exposing Vectors Analytically; Spectral Decomposition

We use the following simple result to obtain the exposing vectors of the minimal face in the problem analytically, i.e., we find the matrices V with orthonormal columns.

Lemma (analytic FR)

Let $\mathcal{C} \subseteq \mathbb{H}^n_+$ be a given closed convex set with nonempty interior. Let $Q_i \in \mathbb{H}^{k\times n}$, $i = 1, \ldots, t$, be given matrices. Define the linear map $\mathcal{A} : \mathbb{H}^n \to \mathbb{H}^k$ and matrix V by

$$\mathcal{A}(X) = \sum_i^t Q_i X Q_i^*, \quad \operatorname{range}(V) = \operatorname{range}\left(\sum_{i=1}^t Q_i Q_i^*\right).$$

Then the minimal face,

$$\operatorname{face}(\mathcal{A}(\mathcal{C})) = V\mathbb{H}^r_+ V^*.$$

Obtain the Minimal Faces, $V\mathbb{H}_+V^*$

$$
\begin{array}{rcl}
\rho & = & V_\rho R_\rho V_\rho^* \in \mathbb{H}_+^n, \quad R_\rho \in \mathbb{H}_+^{n_\rho} \\
\delta & = & V_\delta R_\delta V_\delta^* \in \mathbb{H}_+^k, \quad R_\delta \in \mathbb{H}_+^{k_\delta} \\
\sigma & = & V_\sigma R_\sigma V_\sigma^* \in \mathbb{H}_+^k, \quad R_\sigma \in \mathbb{H}_+^{k_\sigma}
\end{array}
$$

Define the linear maps

$$
\begin{array}{llcrcl}
\Gamma_V: & \mathbb{H}_+^{n_\rho} \to \mathbb{R}^m & \text{by} & \Gamma_V(R_\rho) & = & \Gamma(V_\rho R_\rho V_\rho^*), \\
\mathcal{G}_V: & \mathbb{H}_+^{n_\rho} \to \mathbb{H}_+^k & \text{by} & \mathcal{G}_V(R_\rho) & = & \mathcal{G}(V_\rho R_\rho V_\rho^*), \\
\mathcal{Z}_V: & \mathbb{H}_+^{k_\delta} \to \mathbb{H}_+^k & \text{by} & \mathcal{Z}_V(R_\delta) & = & \mathcal{Z}(V_\delta R_\delta V_\delta^*).
\end{array}
$$

Substitute with Linear Mapping $\mathcal{V}_\delta(\cdot) := V_\delta \cdot V_\delta^*$

Equivalent Formulation

$$
\begin{array}{ll}
\min & \operatorname{trace}(R_\delta \log(R_\delta)) - \operatorname{trace}\left(R_\sigma \log(R_\sigma)\right) \\
\text{subject to:} & \Gamma_V(R_\rho) = \gamma \\
& \mathcal{V}_\sigma(R_\sigma) - \mathcal{Z}_V(R_\delta) = 0 \\
& \mathcal{V}_\delta(R_\delta) - \mathcal{G}_V(R_\rho) = 0 \\
& R_\rho, R_\sigma, R_\delta \succeq 0.
\end{array}
$$

After Rotation and Substitution; final model (QKD)

$$
\begin{array}{rll}
p^* = & \min & f(\rho) = \operatorname{trace}\left(\widehat{\mathcal{G}}(\rho)(\log \widehat{\mathcal{G}}(\rho))\right) \\
& & \qquad - \operatorname{trace}\left(\widehat{\mathcal{Z}}(\rho) \log \widehat{\mathcal{Z}}(\rho)\right) \\
& \text{subject to:} & \Gamma_V(\rho) = \gamma_V \\
& & \rho \in \mathbb{H}_+^{n_\rho},
\end{array}
$$

Slater holds; smaller regularized problem;
positive definiteness preserved in obj. fn

Derivatives

Theorem (Derivatives of regularized objective)

Let $\rho \succ 0$*. The gradient of f is*

$$\nabla f(\rho) = \boxed{\widehat{\mathcal{G}}^*(\log[\widehat{\mathcal{G}}(\rho)]) + \widehat{\mathcal{G}}^*(I)} - \boxed{\widehat{\mathcal{Z}}^*(\log[\widehat{\mathcal{Z}}(\rho)]) + \widehat{\mathcal{Z}}^*(I)}.$$

The Hessian in direction $\Delta\rho$ *is (1st order info)*

$$\begin{aligned}\nabla^2 f(\rho)(\Delta\rho) &= \boxed{\widehat{\mathcal{G}}^*(\log'[\widehat{\mathcal{G}}(\rho)](\widehat{\mathcal{G}}(\Delta\rho))} - \\ &\quad \boxed{\widehat{\mathcal{Z}}^*(\log'[\widehat{\mathcal{Z}}(\rho)](\widehat{\mathcal{Z}}(\Delta\rho))}\end{aligned}$$

Theorem (subdifferential)

Let $\{\rho_i\}_i \subseteq \mathbb{S}^{n_\rho}_{++}$ *with* $\rho_i \to \bar{\rho}$*. If we have the convergence* $\lim_i \nabla f(\rho_i) = \phi$*, then*

$$\phi \in \partial f(\bar{\rho}).$$

Part II: Opt. Cond.; Bounds; GN Int. Pt. Method

- Duality, and primal-dual optimality conditions with null-space representation
- Derive a Gauss-Newton search direction for the nonlinear SDP (with exact primal and dual feasibility possible)
- derive provable lower and upper bounds
- Empirics

Facially Reduced (Regularized) Nonlinear SDP

$$\begin{array}{rccl} p^* = & \min & f(\rho) & \text{(regularized relative entropy)} \\ & \text{subject to:} & \Gamma_V(\rho) = \gamma_V & \text{(FR constraints)} \\ & & \rho \in \mathbb{H}_+^{n_\rho} & \text{(smaller SDP constr.)} \end{array}$$

Duality/Optimality

Theorem (Basic Duality/Opt)

1. *Lagrangian* $L(\rho, y) = f(\rho) + \langle y, \Gamma_V \rho - \gamma_V \rangle$, $y \in \mathbb{R}^{m_V}$.
2. *Strong Duality*

$$\begin{aligned} p^* &= \max_y \min_{\rho \succeq 0} L(\rho, y) \\ &= d^* = \max_{Z \succeq 0, y} \left(\min_\rho (L(\rho, y) - \langle Z, \rho \rangle) \right) \end{aligned}$$

and d^* *is attained for some* $(y, Z) \in \mathbb{R}^{m_V} \times \mathbb{H}_+^{n_\rho}$.

3. *p-d pair* $(\rho, (y, Z))$, *with* $\partial f(\rho) \neq \emptyset$, *is optimal iff*

$$\begin{array}{rcll} 0 & \in & \partial f(\rho) + \Gamma_V{}^* y - Z & \text{(dual feasibility)} \\ 0 & = & \Gamma_V \rho - \gamma_V & \text{(linear primal feasibility)} \\ 0 & = & \langle \rho, Z \rangle & \text{(complementary slackness)} \\ 0 & \preceq & \rho, Z & \text{(SDP primal feasibility).} \end{array}$$

Moreover, $\Gamma_V^* y \succeq 0$, $\langle y, \gamma_V \rangle < 0$, *for some* y, *implies infeas.*

Nullspace Representation/Residuals

Definition (nullspace representation)

$\hat{\rho} \in \mathbb{H}^{n_\rho}$ feasible point for $\Gamma_V(\cdot) = \gamma_V$.
$\mathcal{N}^* : \mathbb{R}^{n_\rho^2 - m_V} \to \mathbb{H}^{n_\rho}$ injective linear map in adjoint form so that we have the nullspace representation for the residual:

$$F_\mu^p = \Gamma_V \rho - \gamma_V \iff F_\mu^p = \mathcal{N}^*(v) + \hat{\rho} - \rho, \text{ for some } v.$$

Perturbed Optimality Conditions/Residuals

(i) dual feas.; (ii) primal feas.; (iii) perturbed compl. slack.

$$F_\mu(\rho, v, y, Z) = \begin{bmatrix} F_\mu^d \\ F_\mu^p \\ F_\mu^c \end{bmatrix} = \begin{bmatrix} \nabla_\rho f(\rho) + {\Gamma_V}^* y - Z \\ \mathcal{N}^* v + \hat{\rho} - \rho \\ Z\rho - \mu I \end{bmatrix} = 0, \quad \rho, Z \succ 0.$$

EXACT p-d feas if updated as: $\rho \leftarrow \Delta v$; $Z \leftarrow \Delta\rho, \Delta y$;
after a steplength $= 1$ is taken, exact p.f. is maintained.
exact dual feas. for Z is key for lower bound

Projected Gauss-Newton P-D I-P Method

Linearized System for GN Direction; OVERDETERMINED

$$F'_\mu d_{GN} = \begin{bmatrix} \nabla^2 f(\rho)\Delta\rho + \Gamma_V{}^*\Delta y - \Delta Z \\ \mathcal{N}^*(\Delta v) - \Delta\rho \\ Z\Delta\rho + \Delta Z\rho \end{bmatrix} \approx -F_\mu.$$

From First Block (for backsubstitution)

$$\begin{aligned} \Delta Z &= F_\mu^d + \nabla^2 f(\rho)\Delta\rho + \Gamma_V{}^*\Delta y \\ &= F_\mu^d + \nabla^2 f(\rho)(F_\mu^p + \mathcal{N}^*(\Delta v)) + \Gamma_V{}^*\Delta y. \end{aligned}$$

From Second Block (for backsubstitution)

$$\Delta\rho = F_\mu^p + \mathcal{N}^*(\Delta v).$$

Now substitute $\Delta Z, \Delta\rho$ into third block.

Projected GN direction

$d_{GN} = (\Delta v \quad \Delta y)$ (backsubst. for $\Delta\rho, \Delta Z$)

found from the least squares solution of (OVERDETERMINED)

$$\left[Z\mathcal{N}^*(\Delta v) + \nabla^2 f(\rho)\mathcal{N}^*(\Delta v)\rho\right] + [\Gamma_V{}^*\Delta y \rho] \\ = -F_\mu^c - ZF_\mu^p - \left(F_\mu^d + \nabla^2 f(\rho) F_\mu^p\right)\rho$$

(Uses Hessian acting on a vector: $\nabla^2 f(\rho) : \mathbb{H} \to \mathbb{H}$)

Initialize: $\hat{\rho} \succ 0$, $\mu \in \mathbb{R}_{++}$, $\eta \in (0,1)$
WHILE: stopping criteria is not met
solve for $(\Delta v, \Delta y)$
$\Delta\rho = F_\mu^p + \mathcal{N}^*(\Delta v)$
$\Delta Z = F_\mu^d + \nabla^2 f(\rho)(F_\mu^p + \mathcal{N}^*(\Delta v)) + \Gamma_V{}^*\Delta y$
choose steplength α
$(\rho, y, Z) \leftarrow (\rho, y, Z) + \alpha(\Delta\rho, \Delta y, \Delta Z)$
$\mu = \text{trace}(\rho Z)/n$; $\mu \leftarrow \eta\mu$
ENDWHILE

Implementation Details

Sparse Nullspace Representation

We use a matrix representation M for Γ and a row and column permutation to get a well-conditioned near triangular basis matrix B. nullspace representation:

$$\hat{r} = \text{Hvec}\,\hat{\rho};\ \Gamma_V\hat{\rho} = M\hat{r}(cp) = \gamma_V,\ M = \begin{bmatrix} B & E \end{bmatrix},\ N^* = \begin{bmatrix} B^{-1}E \\ -I \end{bmatrix};$$

Optimal Diagonal Preconditioning, [4]

$d_i = \|F_\mu^{c\prime}(e_i)\|$, for unit vectors e_i; column precondition using

$$F_\mu^{c\prime} \leftarrow F_\mu^{c\prime}\mathrm{Diag}\,(d)^{-1}$$

MATLAB: $d_{GN} = ((F_\mu^{c\prime}/\mathrm{Diag}\,(d))\backslash RHS)./d$
Performed exceptionally well; problems are VERY ill-conditioned.

Upper Bounds

Evaluate f at Feasible ρ; Iterative Refinement if Needed

if approximate linear feasibility $\Gamma_V \hat{\rho} \approx \gamma_V$, we apply iterative refinement by finding the projection Let $\hat{\rho} \succ 0$, $F^p_\mu = \Gamma_V \hat{\rho} - \gamma_V$. Then

$$\rho = \hat{\rho} - {\Gamma_V}^\dagger F^p_\mu = \operatorname{argmin}_\rho \left\{ \frac{1}{2} \|\rho - \hat{\rho}\|^2 \;:\; \Gamma_V \rho = \gamma_V \right\},$$

where we denote ${\Gamma_V}^\dagger$, generalized inverse. If $\rho \succeq 0$, then $p^* \leq f(\rho)$.

EXACT Primal Feasibility

In our tests we take a Newton step quite early, and maintain exact primal feasibility (no roundoff error buildup) for the further iterations.

Lower Bound from Weak Duality

$\rho \succ 0, Z \succ 0$ in Interior Point Algorithm

The gradients exist at $\rho \succ 0$; we can verify dual feasibility.

Corollary (Lower Bound for FR problem from EXACT Dual Feas.)

$\hat{\rho}, \hat{y}$ primal-dual iterate; $\hat{\rho} \succ 0$. Set $\boxed{\bar{Z} = \nabla f(\hat{\rho}) + \Gamma_V{}^* \hat{y}}$.
If $\bar{Z} \succeq 0$, then lower bound is:

$$p^* \geq f(\hat{\rho}) + \langle \hat{y}, \Gamma_V \hat{\rho} - \gamma_V \rangle - \langle \hat{\rho}, \bar{Z} \rangle.$$

Proof.

Consider the dual problem
$d^* = \max_{y, Z \succeq 0} \min_{\rho \in \mathbb{H}^{n_\rho}} L(\rho, y) - \langle Z, \rho \rangle$. Dual feasibility implies:

$$\bar{Z} \succeq 0,\ \nabla f(\hat{\rho}) + \Gamma_V{}^* \hat{y} - \bar{Z} = 0 \implies \hat{\rho} \in \operatorname{argmin}_\rho L(\rho, \hat{y}) - \langle \bar{Z}, \rho \rangle.$$

Result follows from weak duality. □

Numerics; GN vs FW with/without FR

Problem Data		Gauss-Newton		FW (FR)		FW (no FR)		cvxquad (FR)	
protocol	size	gap	time	gap	time	gap	time	gap	time
ebBB84	(4,16)	6.0e-13	0.4	1.0e-04	86.9	1.2e-04	93.9	5.5e-01	194
ebBB84	(4,16)	1.2e-12	0.3	1.7e-04	96.8	1.3e-04	110.5	5.4e-01	1938
ebBB84	(4,16)	1.1e-12	0.2	1.6e-04	86.5	2.2e-04	112.0	5.7e-01	1979
ebBB84	(4,16)	4.2e-13	0.2	2.2e-04	88.6	2.2e-04	111.9	6.3e-01	523
pmBB84	(8,32)	5.5e-13	0.2	3.1e-05	1.3	6.5e-04	1.6	5.3e-01	158
pmBB84	(8,32)	6.1e-13	0.2	1.6e-04	1.1	3.8e-04	93.0	5.2e-01	207
pmBB84	(8,32)	6.3e-13	0.2	5.5e-05	1.1	3.0e-04	112.3	5.6e-01	299
pmBB84	(8,32)	1.3e-12	0.2	2.6e-04	1.1	1.3e-03	87.0	5.9e-01	188
mdiBB84	(48,96)	7.9e-13	0.9	9.6e-05	1.6	5.4e-04	120.9	1.8e-01	570
mdiBB84	(48,96)	1.4e-12	0.7	5.5e-05	101.2	6.6e-04	119.7	2.4e-01	585
mdiBB84	(48,96)	5.7e-13	0.9	1.5e-04	101.9	1.7e-03	439.3	3.1e-01	584
mdiBB84	(48,96)	9.2e-13	0.8	1.8e-04	100.0	2.2e-03	441.3	3.7e-01	558

Table: Numerical Report: Gauss-Newton, Frank-Wolfe (FW), cvxquad

- GN performs significantly better for both accuracy and running time
- only three protocols (each with four different parameter settings);
  that is all that cvxquad could handle;
- FW is significantly improved by using our new FR

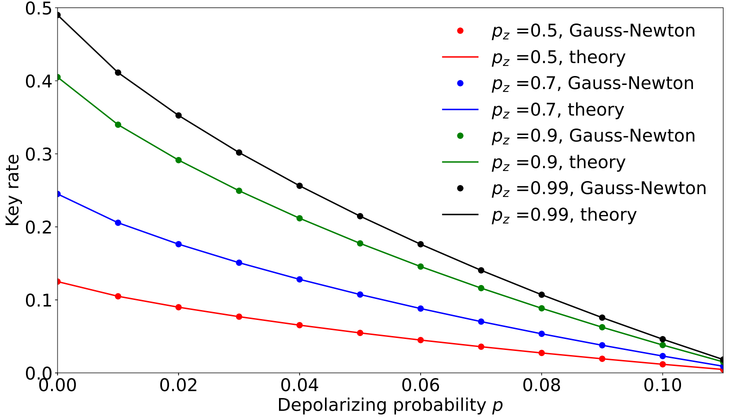
Larger Numerics; cvxquad Failed

Problem Data		Gauss-Newton		FW (FR)		FW (no FR)	
protocol	**size**	**gap**	**time**	**gap**	**time**	**gap**	**time**
TFQKD	(12,24)	5.9e-13	1.1	2.6e-09	1.9	1.6e-03	364.1
TFQKD	(12,24)	1.2e-12	0.8	3.8e-09	1.5	5.6e-04	369.1
TFQKD	(12,24)	3.2e-13	0.8	4.0e-09	1.3	1.7e-04	4.1
DMCV	(44,176)	2.7e-09	1326.1	2.4e-06	2808.4	3.4e-06	4933.9
DMCV	(44,176)	2.7e-09	1377.4	1.3e-06	974.2	2.5e-06	1281.2
DMCV	(48,192)	3.1e-09	1807.1	2.7e-06	3167.4	5.1e-06	5407.5
DMCV	(48,192)	3.2e-09	2110.6	2.6e-06	979.8	2.0e-06	1756.3
dprBB84	(12,48)	4.9e-13	1.3	3.8e-06	88.0	9.4e-05	123.0
dprBB84	(24,96)	1.0e-12	12.1	6.2e-06	15.9	3.6e-06	31.1
dprBB84	(36,144)	5.0e-13	69.3	6.5e-04	8.8	2.1e-02	30.1
dprBB84	(48,192)	1.1e-12	325.5	4.4e-05	17.1	9.8e-04	181.9

Table: Numerical Report: Gauss-Newton, Frank-Wolfe (FW)

- GN performs significantly better again
- FW does significantly better with our new FR again

GN is Exactly Analytical (Protocol mdiBB84)



The • are the lower bounds from GN; they coincide exactly with the analytical values on the curves.
This meets with the empirical evidence of gaps $\approx 10^{-12}$

Conclusion

- regularized the key rate calculation using FACIAL REDUCTION on both constraints and nonlinear (relative entropy) objective function over the Hermitians (complex);
- provided theoretically proven upper and lower bounds with high precision
- derived robust (Gauss-Newton) interior point approach on regularized problem
  - avoids current perturbation approach to get $\rho \succ 0$;
  - avoids roundoff error from backsubstitution steps;
  - attains exact primal feasibility during iterations
  - uses exact dual feasibility steps to improve on lower bounds

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Thanks for your attention!

Robust Interior Point Methods and FR for Key Rate Computation in Quantum Key Distribution

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