Compositions of projection mappings: fixed point sets and difference vectors

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Abstract

Projection operators and associated projection algorithms are fundamental building blocks in fixed point theory and optimization.

In this talk, I will survey recent results on the displacement mapping of the right-shift operator and sketch a new application deepening our understanding of the geometry of the fixed point set of the composition of projection operators in Hilbert space.

Based on joint works with Salha Alwadani, Julian Revalski, and Shawn Wang.

Acknowledgments

First and foremost, a

BIG THANK YOU

to the organizers for keeping us optimistically optimizing during the past 14 months!!

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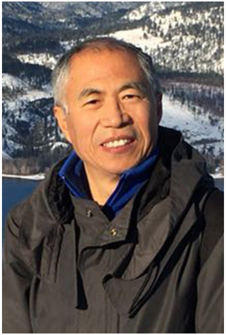
This talk is based on joint works with various collaborators, most recently with

- Salihah Alwadani (UBC Okanagan)
- Julian Revalski (Academy of Sciences, Bulgaria)
- Xianfu Wang (UBC Okanagan)

Salha Alwadani

Julian Revalski



Shawn Wang

Introduction

The setting

Throughout, suppose that

$$X \text{ is a real Hilbert space,}$$

with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and $m \geq 2$. Assume that

$$C_1, C_2, \ldots, C_m \quad \text{are nonempty closed convex subsets of } X,$$

with associated projectors (nearest-point mappings)

$$\mathrm{P}_1 := \mathrm{P}_{C_1}, \ldots, \mathrm{P}_m := \mathrm{P}_{C_m}.$$

Our goal is to understand the fixed point set

$$F_m := \mathrm{Fix}(\mathrm{P}_m \cdots \mathrm{P}_2 \mathrm{P}_1).$$

We also define cyclically

$$\begin{aligned} F_{m-1} &:= \mathrm{Fix}(\mathrm{P}_{m-1}\mathrm{P}_{m-2} \cdots \mathrm{P}_1 \mathrm{P}_m) \\ &\vdots \\ F_1 &:= \mathrm{Fix}(\mathrm{P}_1 \mathrm{P}_m \cdots \mathrm{P}_3 \mathrm{P}_2). \end{aligned}$$

Why care?

A significant number of convex optimization problems are convex feasibility problems of the form

$$\text{Find } x \in C_1 \cap C_2 \cap \cdots \cap C_m =: S.$$

Very beautifully,

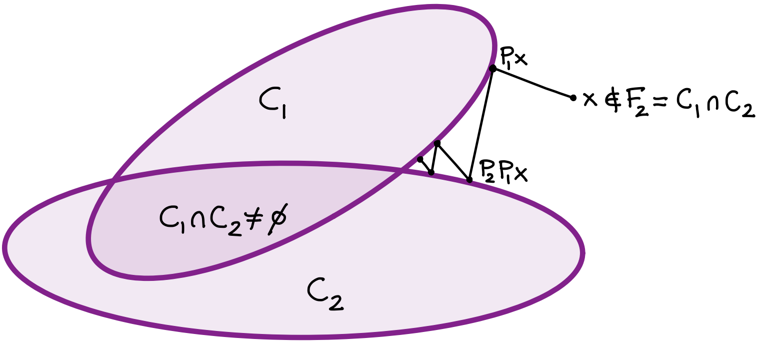
$$S \neq \varnothing \quad \Rightarrow \quad S = F_m = F_{m-1} = \cdots = F_1.$$

And, very usefully, when $S \neq \varnothing$, then a point in S can be found as the weak limit of the sequence

$$\begin{aligned} x_0, &\mathrm{P}_1 x_0, \mathrm{P}_2 \mathrm{P}_1 x_0, \ldots, \mathrm{P}_m \cdots \mathrm{P}_1 x_0 \\ &\mathrm{P}_1 \mathrm{P}_m \cdots \mathrm{P}_1 x_0, \ldots, (\mathrm{P}_m \cdots \mathrm{P}_1)^2 x_0, \\ &\mathrm{P}_1 (\mathrm{P}_m \cdots \mathrm{P}_1)^2 x_0, \ldots, (\mathrm{P}_m \cdots \mathrm{P}_1)^3 x_0, \ldots \end{aligned}$$

generated by the method of cyclic projections a.k.a. "POCS" (**P**rojections **O**nto **C**onvex **S**ets).
This was applied by Sir Godfrey Hounsfield who won a Nobel prize in 1979 for his work on Computer-assisted Tomography.

$S \neq \varnothing \Rightarrow S = F_m = \cdots = F_1$



When $S = C_1 \cap \cdots \cap C_m \neq \varnothing$, then all fixed point sets F_i must coincide with S.

The question

Question: What happens when $S = \varnothing$? What is F_m??

$m = 2$

Suppose now (temporarily) that $m = 2$.
Consider the "gap" between C_1 and C_2:

$$\delta := \inf_{(c_1,c_2)\in C_1\times C_2} \|c_1 - c_2\|.$$

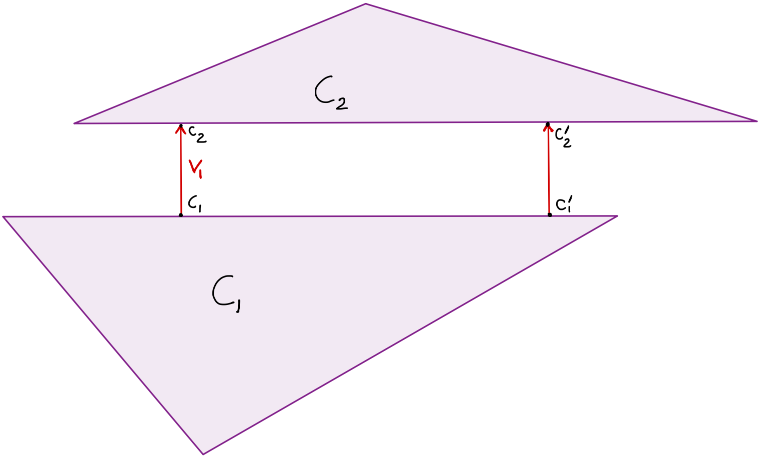
This is an *infimum* which may or may not be attained.
Let's assume δ is actually attained, i.e., *the infimum is a minimum*.
Then the following hold for (c_1, c_2) and (c_1', c_2') in $C_1 \times C_2$:

- $\|c_1 - c_2\| = \delta \Leftrightarrow c_2 = \mathrm{P}_2 c_1$ and $c_1 = \mathrm{P}_1 c_2$ in which case (c_1, c_2) form a *best approximation pair*
- $\|c_1 - c_2\| = \delta \Rightarrow c_2 \in F_2$ and $c_1 \in F_1$
- F_1 and F_2 are both nonempty
- $\|c_1 - c_2\| = \delta$ and $\|c_1' - c_2'\| = \delta$ implies

$$v_1 := c_2 - c_1 = c_2' - c_1',$$

  the difference (a.k.a. gap) vector is well defined!

$m = 2$ and δ infimum attained



Two best approximation pairs (c_1, c_2) and (c_1', c_2')

$m = 2$ (continued)

Recall $\delta := \inf_{(c_1,c_2)\in C_1\times C_2} \|c_1 - c_2\|$.

If one of the sets is bounded, say C_2, then δ is always attained.

Sketch of Proof:

- The map
$$\mathrm{P}_2\mathrm{P}_1 \colon C_2 \to C_2$$
is a nonexpansive (Lipschitz-1) selfmap of C_2.
- By the *Browder–Göhde–Kirk Fixed Point Theorem*, this map has a fixed point $\bar{x} \in C_2$:
$$\bar{x} = \mathrm{P}_2\mathrm{P}_1\bar{x}.$$
- But then
$$(c_1, c_2) = \big(\mathrm{P}_1\bar{x}, \bar{x}\big)$$
forms a best approximation pair and attains the infimum!

Remark: This proof also works for $m \geq 2$!

$m = 2$ (continued)

Recall $\delta := \inf_{(c_1,c_2)\in C_1\times C_2} \|c_1 - c_2\|$.

However, it can happen that δ is not attained:

Example. Assume that in $X = \mathbb{R}^2$ we have

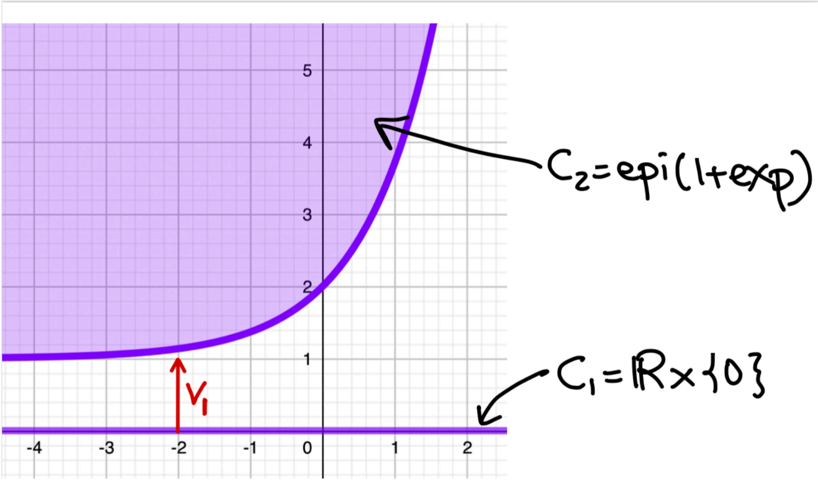
$$C_1 = \mathbb{R} \times \{0\},$$
$$C_2 = \operatorname{epi}(1 + \exp) = \{(\xi, \eta) \in \mathbb{R}^2 \mid 1 + \exp(\xi) \le \eta\}.$$

Then $\delta = 1$, but the infimum is not attained.

Example. Suppose that C_1 and C_2 are closed affine subspaces, say $C_1 = c_1 + L_1,\ C_2 = c_2 + L_2$, where $L_1,\ L_2$ are linear subspaces. Then:

- If X is finite-dimensional, then δ is always attained.
- If X is infinite-dimensional, then one can construct C_1 and C_2 such that the Minkowski sum
$$C_1 + C_2 \text{ is not closed}$$
and δ is not attained!

$m = 2$ and δ infimum not attained



The (magic) difference vectors

Whether or not δ is attained, we can always (well!) define the so-called difference vectors

$$v_1 := P_{\overline{C_2 - C_1}}(0),$$
$$v_2 := P_{\overline{C_1 - C_2}}(0).$$

(Here $C_2 - C_1 = \{c_2 - c_1 \mid (c_1, c_2) \in C_1 \times C_2\}$.) These satisfy

$$\|v_1\| = \|v_2\| = \delta \text{ and } v_1 + v_2 = 0.$$

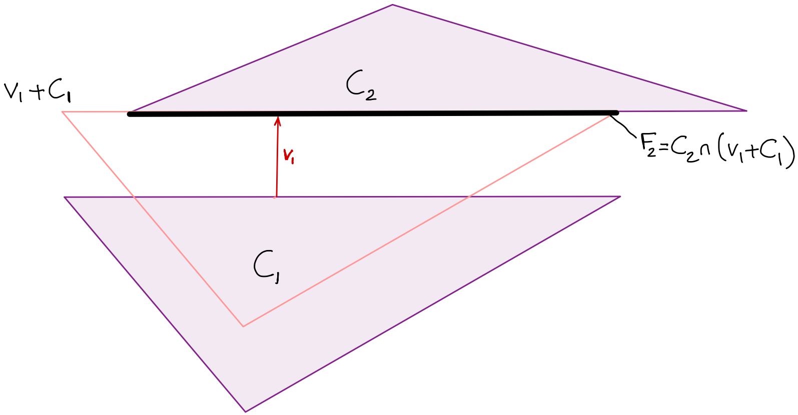
Moreover, the difference vectors satisfy

$$F_2 = C_2 \cap (v_1 + C_1)$$
$$F_1 = C_1 \cap (v_2 + C_2).$$

In other words, the sets F_1 and F_2 are obtained in two easy steps:

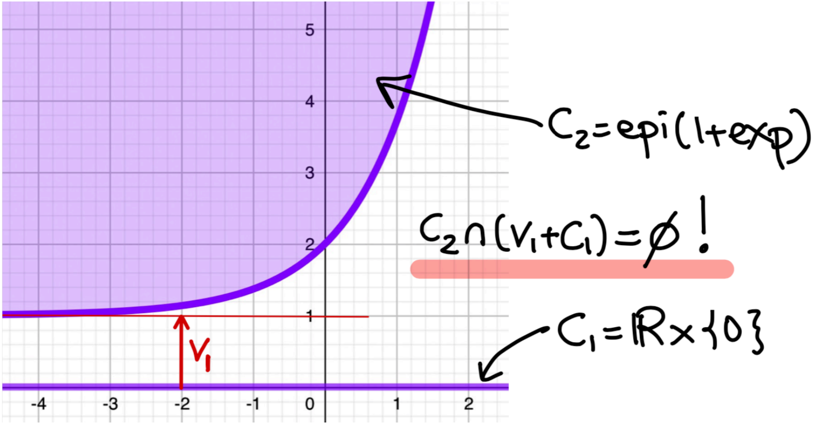
Step 1 translate/shift the original sets C_1 and C_2 using v_1 and v_2,
Step 2 then intersect!

The difference vector for $m = 2$ and δ attained



δ attained and $F_2 = C_2 \cap (v_1 + C_1) \neq \varnothing$

The difference vector for $m = 2$ and δ unattained



The geometry conjecture

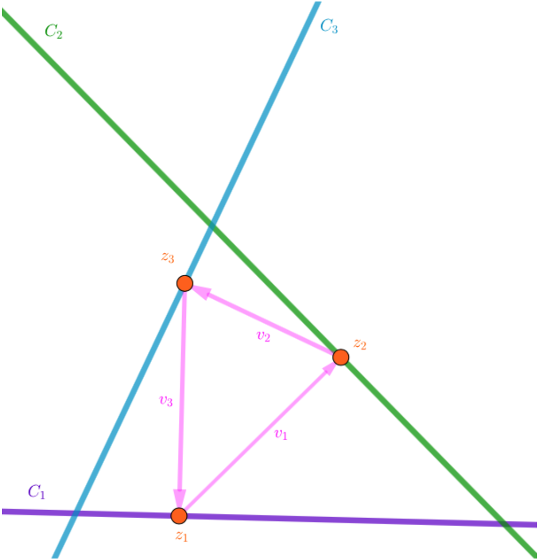
The geometry conjecture, first posed in 1997 by Bauschke, Borwein, and Lewis (*Contemp. Math.*), states the following:

> *Suppose $m \geq 2$. Then there exist vectors $v_1, v_2, \ldots, v_m$ such that*
>
> $$\begin{aligned} F_m = C_m &\cap (v_{m-1} + C_{m-1}) \\ &\cap (v_{m-1} + v_{m-2} + C_{m-2}) \\ &\qquad\vdots \\ &\cap (v_{m-1} + \cdots + v_2 + v_1 + C_1) \end{aligned}$$
>
> *and similarly for $F_{m-1}, \ldots, F_1$.*

When $\bigcap_{i=1}^m C_i \neq \varnothing$ ($\Rightarrow v_1 = \cdots v_m = 0$), or when $m = 2$, the geometry conjecture holds true by works of Cheney and Goldstein (*Proc. AMS* 1959) and of Bauschke and Borwein (*JAT* 1994).

What happens when $m \geq 3$?

The problem in one picture



What are the difference vectors v_1, v_2, v_3?

Cycles

Suppose that $F_m \neq \varnothing$, and pick $z_m \in F_m$, i.e.,

$$z_m = \mathsf{P}_m \cdots \mathsf{P}_2 \mathsf{P}_1 z_m.$$

Then

$$\mathsf{P}_1 z_m = \mathsf{P}_1 \mathsf{P}_m \cdots \mathsf{P}_2 \mathsf{P}_1 z_m = \big(\mathsf{P}_1 \mathsf{P}_m \cdots \mathsf{P}_2\big)\big(\mathsf{P}_1 z_m\big)$$

and so

$$\begin{aligned} z_1 &:= \mathsf{P}_1 z_m \in F_1 \\ z_2 &:= \mathsf{P}_2 z_1 \in F_2 \\ &\vdots \\ z_{m-1} &:= \mathsf{P}_{m-1} z_{m-2} \in F_{m-1} \\ z_m &= \mathsf{P}_m z_{m-1}. \end{aligned}$$

We refer to the vector $\mathbf{z} := (z_1, z_2, \ldots, z_m) \in X^m$ as a cycle.

$$\text{Cycles exist} \Leftrightarrow F_m \neq \varnothing.$$

Cycles and optimization!

If $m = 2$, then $\mathrm{z} = (z_1, z_2)$ is a cycle if and only if z is a minimizer of the function

$$\mathrm{x} = (x_1, x_2) \mapsto \iota_{C_1}(x_1) + \iota_{C_2}(x_2) + \tfrac{1}{2}\|x_1 - x_2\|^2.$$

This suggests the conjecture that $\mathrm{z} = (z_1, \dots, z_m)$ is a cycle if it minimizes a function of the form

$$\mathrm{x} = (x_1, \dots, x_m) \mapsto \sum_{i=1}^{m} \iota_{C_i}(x_i) + \varphi(x_1, \dots, x_m);$$

however, Baillon, Combettes, and Cominetti proved in 2012 that this is *impossible* when $m \geq 3$: *cycles defy optimization!*

On the positive side, we can deduce some information about the difference vectors and the geometry conjecture when we have cycles!

Cycles and difference vectors

Assume that we have two cycles, say $\mathrm{z} := (z_1, z_2, \ldots, z_m)$ and $\mathrm{y} := (y_1, y_2, \ldots, y_m)$.

For notational simplicity, let's assume that $m = 3$ (the proof easily extends to general m).

Then, by definition of "cycle",

$$\begin{aligned} z_1 &= \mathrm{P}_1 z_3 \\ z_2 &= \mathrm{P}_2 z_1 \\ z_3 &= \mathrm{P}_3 z_2 \end{aligned}$$

and similarly for (y_1, y_2, y_3).

Because projectors are *firmly nonexpansive*, this implies ...

Cycles and difference vectors (continued)

$$
\begin{aligned}
\|y_3 - z_3\|^2 &= \|\mathrm{P}_3 y_2 - \mathrm{P}_3 z_2\|^2 \\
&\leq \|y_2 - z_2\|^2 - \|(\mathrm{Id} - \mathrm{P}_3)y_2 - (\mathrm{Id} - \mathrm{P}_3)z_2\|^2 \\
&\leq \|y_1 - z_1\|^2 - \|(\mathrm{Id} - \mathrm{P}_2)y_1 - (\mathrm{Id} - \mathrm{P}_2)z_1\|^2 \\
&\qquad - \|(\mathrm{Id} - \mathrm{P}_3)y_2 - (\mathrm{Id} - \mathrm{P}_3)z_2\|^2 \\
&\leq \|y_3 - z_3\|^2 - \|(\mathrm{Id} - \mathrm{P}_1)y_3 - (\mathrm{Id} - \mathrm{P}_1)z_3\|^2 \\
&\qquad - \|(\mathrm{Id} - \mathrm{P}_2)y_1 - (\mathrm{Id} - \mathrm{P}_2)z_1\|^2 \\
&\qquad - \|(\mathrm{Id} - \mathrm{P}_3)y_2 - (\mathrm{Id} - \mathrm{P}_3)z_2\|^2.
\end{aligned}
$$

Hence *equality* holds throughout, which implies that *all nonpositive terms are equal to zero* and thus:

$$
\begin{aligned}
y_2 - \mathrm{P}_3 y_2 = z_2 - \mathrm{P}_3 z_2 &\Leftrightarrow y_2 - y_3 = z_2 - z_3 \\
y_1 - \mathrm{P}_2 y_1 = z_1 - \mathrm{P}_2 z_1 &\Leftrightarrow y_1 - y_2 = z_1 - z_2; \\
y_3 - \mathrm{P}_1 y_3 = z_3 - \mathrm{P}_1 z_3 &\Leftrightarrow y_3 - y_1 = z_3 - z_1.
\end{aligned}
$$

Cycles and difference vectors (continued)

We thus *well* define

$$v_1 := z_2 - z_1$$
$$v_2 := z_3 - z_2$$
$$v_3 := z_1 - z_3$$

where $z = (z_1, z_2, z_3)$ is *any* cycle. Then

$$z_3 \in C_3$$
$$z_3 = v_2 + z_2 \in v_2 + C_2$$
$$z_3 = v_2 + v_1 + z_1 \in v_2 + v_1 + C_1.$$

We thus proved

$$F_3 \subseteq C_3 \cap (v_2 + C_2) \cap (v_2 + v_1 + C_1),$$

which is one half of the geometry conjecture! But:

$\ominus$ the other half of the geometry conjecture is still missing;

$\ominus$ we don't know how to define the v_i *intrinsically*,
i.e., in the absence of cycles, when $m \geq 3$!

Geometry conjecture

Regarding the geometry conjecture, we know:

- it is true when $C_1 \cap \cdots \cap C_m \neq \varnothing$ (with $v_i \equiv 0$);
- it is true when $m = 2$;
- one half is known to be true when there are cycles.

GOOD NEWS: The geometry conjecture is true in general !!

Main Result and Tools utilized

Product space

We will work in the product space

$$\mathsf{X} := X^m$$

in which we set

$$\mathsf{C} := C_1 \times \cdots \times C_m \;\text{ and }\; \Delta := \big\{(x, \ldots, x) \in \mathsf{X} \;\big|\; x \in X\big\}.$$

It is well known that

$$\mathsf{P}_{\mathsf{C}}(x_1, \ldots, x_m) = \big(\mathsf{P}_1 x_1, \ldots, \mathsf{P}_m x_m\big)$$

and

$$\mathsf{P}_{\Delta}(x_1, \ldots, x_m) = \frac{1}{m}\Big(\sum_{i=1}^{m} x_i, \ldots, \sum_{i=1}^{m} x_i\Big).$$

The right shift operator

Define the (circular) right shift operator R by

$$\mathrm{R}\colon \mathrm{X} \to \mathrm{X}\colon (x_1, x_2, \ldots, x_m) \mapsto (x_m, x_1, x_2, \ldots, x_{m-1}).$$

Then

$$\mathrm{z} = (z_1, \ldots, z_m) \text{ is a cycle } \Leftrightarrow \mathrm{z} = \mathrm{P_C}(\mathrm{Rz}).$$

Denote the (possibly empty) set of all cycles by

$$\mathrm{Z} := \mathrm{Fix}(\mathrm{P_C R}).$$

Now $\mathrm{z} = \mathrm{P_C Rz} = (\mathrm{Id} + N_\mathrm{C})^{-1}\mathrm{Rz} \Leftrightarrow \mathrm{Rz} \in \mathrm{z} + N_\mathrm{C}\mathrm{z} \Leftrightarrow$

$$0 \in N_\mathrm{C}(\mathrm{z}) + (\mathrm{Id} - \mathrm{R})(\mathrm{z}), \tag{P}$$

which we view as a (primal) sum problem!

Note that the operator sum in (P) is maximally monotone.

Recall that $A\colon X \rightrightarrows X$ is monotone if $(\forall x^* \in Ax)(\forall y^* \in Ay)$ $\langle x - y, x^* - y^* \rangle \geq 0$; it is maximally monotone if it cannot be enlarged without destroying monotonicity: e.g., subdifferentials of convex functions and matrices whose symmetric part is $\succeq 0$.

The (Attouch–Théra) dual and the skew operator T

The Attouch–Théra dual of (P) is

$$0 \in N_C^{-1}(y) + (\mathrm{Id} - R)^{-1}(y). \tag{D}$$

This operator sum is not necessarily maximally monotone!
If y solves (D), then

$$Z = N_C^{-1}(y) \cap -(\mathrm{Id} - R)^{-1}(y) \neq \varnothing.$$

But what is $(\mathrm{Id} - R)^{-1}$? It turns out that

$$(\mathrm{Id} - R)^{-1} = \tfrac{1}{2}\,\mathrm{Id} + N_{\Delta^\perp} + T,$$

where $T^* = -T$ is the skew linear ($\Rightarrow$ monotone) operator

$$\boxed{T := \frac{1}{2m}\sum_{k=1}^{m-1}(m-2k)R^k.}$$

This inversion formula is a consequence of recent work by Alwadani-Bauschke-Revalski-Wang 2020.

Instances of T for $m \in \{2, 3, 4, 5, 6\}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{1}{6}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \frac{1}{4}\begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix},$$

$$\frac{1}{10}\begin{pmatrix} 0 & -3 & -1 & 1 & 3 \\ 3 & 0 & -3 & -1 & 1 \\ 1 & 3 & 0 & -3 & -1 \\ -1 & 1 & 3 & 0 & -3 \\ -3 & -1 & 1 & 3 & 0 \end{pmatrix}, \quad \frac{1}{6}\begin{pmatrix} 0 & -2 & -1 & 0 & 1 & 2 \\ 2 & 0 & -2 & -1 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 & -1 \\ -1 & 0 & 1 & 2 & 0 & -2 \\ -2 & -1 & 0 & 1 & 2 & 0 \end{pmatrix}$$

Only when $m = 2$ do we get a symmetric matrix!

The extended dual and the magic vector y

Using the formula for $(\mathrm{Id} - R)^{-1}$, we rewrite (D) as

$$0 \in N_{\mathsf{C}}^{-1}(\mathsf{y}) + \tfrac{1}{2}\mathsf{y} + N_{\Delta^\perp}(\mathsf{y}) + \mathsf{T}\mathsf{y}. \tag{D}$$

On the other hand,

$$N_{\mathsf{C}}^{-1} + N_{\Delta^\perp} = N_{\mathsf{C}}^{-1} + N_{\Delta}^{-1} = \partial\sigma_{\mathsf{C}} + \partial\sigma_{\Delta} \subseteq \partial(\sigma_{\mathsf{C}} + \sigma_{\Delta}) = \partial\sigma_{\mathsf{C}+\Delta},$$

and this last term is maximally monotone!
Altogether, we consider the following problem which *extends* (D):

$$\boxed{0 \in \tfrac{1}{2}\mathsf{y} + \mathsf{T}\mathsf{y} + \partial\sigma_{\mathsf{C}+\Delta}(\mathsf{y}).} \tag{$\overline{\mathrm{D}}$}$$

Now we have a miracle: $(\overline{\mathrm{D}})$ *always has a unique solution*; in fact, y is the resolvent of the maximally monotone operator

$$2\mathsf{T} + 2\partial\sigma_{\mathsf{C}+\Delta}$$

evaluated at 0! Moreover, one can show that the magic vector y satisfying $(\overline{\mathrm{D}})$ is characterized to be the unique solution to

$$\mathsf{y} \in \Delta^\perp, \;\; -\tfrac{1}{2}\mathsf{y} - \mathsf{T}\mathsf{y} \in \overline{\mathsf{C} + \Delta}, \;\; \text{and} \;\; \sigma_{\mathsf{C}}(\mathsf{y}) + \tfrac{1}{2}\|\mathsf{y}\|^2 \le 0.$$

Main result and the magic vectors e, v

Let y be the unique solution to $(\overline{\mathrm{D}})$.
Set

$$\boxed{\mathrm{e} := -\tfrac{1}{2}\mathrm{y} - \mathrm{Ty} \in \Delta^{\perp}.}$$

Theorem. (Alwadani-Bauschke-Revalski-Wang 2020)

$$\boxed{\mathrm{Z} = \mathrm{e} + \big(\Delta \cap (\mathrm{C} - \mathrm{e})\big).}$$

Then $\mathrm{Z} \neq \varnothing \Leftrightarrow \mathrm{e} \in \mathrm{C} + \Delta$; if $\mathrm{e} = \mathrm{c} + \mathrm{d} \in \mathrm{C} + \Delta$, then $\mathrm{c} \in \mathrm{Z}$.
Finally, set

$$\boxed{\mathrm{v} := -\mathrm{R}^*\mathrm{y} = \mathrm{R}^*\mathrm{e} - \mathrm{e} \in \Delta^{\perp},}$$

where R^*, the adjoint of R, is the (circular) left shift operator.
Then this vector

$$\mathrm{v} = (v_1, \ldots, v_m)$$

is the sought-after difference vector making the geometry conjecture true!!

Revisiting $m = 2$

Difference vectors revisited

It was long known that

$$\mathsf{v} = \left(\mathsf{P}_{\overline{C_2 - C_1}}(0), \mathsf{P}_{\overline{C_1 - C_2}}(0)\right).$$

Our analysis simplifies a lot when $m = 2$; in particular, the skew operator T turns into the zero operator.

This allows us to obtain the alternative description

$$\mathsf{v} = 2P_{\overline{\Delta - \mathsf{C}}}(0),$$

which appears to be new.

Two lines

Let's assume that $m = 2$ and

$$C_1 = c_1 + \mathbb{R}u_1,\ C_2 = c_2 + \mathbb{R}u_2,$$

where

$$c_1 \perp u_1,\ c_2 \perp u_2,\ \text{and } \|u_1\| = \|u_2\| = 1.$$

The parallel case: WLOG $u_1 = u_2 = u$. Then

$$\mathrm{Z} = (c_1, c_2) + \mathbb{R}(u, u),\ \mathrm{v} = (c_2 - c_1, c_1 - c_2) = \mathrm{y} \text{ and } \mathrm{e} = -\tfrac{1}{2}\mathrm{v}.$$

The nonparallel case: Set

$$\rho_1 := \frac{\langle u_1, c_2\rangle + \langle u_1, u_2\rangle\langle u_2, c_1\rangle}{1 - \langle u_1, u_2\rangle^2},\ \rho_2 := \frac{\langle u_2, c_1\rangle + \langle u_1, u_2\rangle\langle u_1, c_2\rangle}{1 - \langle u_1, u_2\rangle^2}$$

Then $\mathrm{Z} = \{\mathrm{z}\}$ is a singleton,

$$\mathrm{z} = (z_1, z_2) = (c_1 + \rho_1 u_1, c_2 + \rho_2 u_2);$$

and

$$\mathrm{v} = (z_2 - z_1, z_1 - z_2) = \mathrm{y} \text{ and } \mathrm{e} = -\tfrac{1}{2}\mathrm{v}.$$

$$m = 3$$

A characterization of the magic vector y

The magic vector $\mathrm{y} = (y_1, y_2, y_3)$ is characterized by the following:

$$y_1 + y_2 + y_3 = 0,$$

there exist sequences $(c_{1,n})_{n\in\mathbb{N}}$ in C_1, $(c_{2,n})_{n\in\mathbb{N}}$ in C_2, $(c_{3,n})_{n\in\mathbb{N}}$ in C_3, and $(x_n)_{n\in\mathbb{N}}$ in X such that

$$\begin{aligned}
c_{1,n} + x_n &\to \tfrac{1}{6}\big(-3y_1 + y_2 - y_3\big),\\
c_{2,n} + x_n &\to \tfrac{1}{6}\big(-y_1 - 3y_2 + y_3\big),\\
c_{3,n} + x_n &\to \tfrac{1}{6}\big(y_1 - y_2 - 3y_3\big),
\end{aligned}$$

and $(\forall (c_1, c_2, c_3) \in C_1 \times C_2 \times C_3)$

$$\langle y_1, c_1\rangle + \langle y_2, c_2\rangle + \langle y_3, c_3\rangle \leq -\tfrac{1}{2}\big(\|y_1\|^2 + \|y_2\|^2 + \|y_3\|^2\big).$$

The magic vectors e and v

Given the magic vector $\mathrm{y} = (y_1, y_2, y_3)$, we have

$$\mathrm{e} = (e_1, e_2, e_3) = \tfrac{1}{6}\big(-3y_1 + y_2 - y_3, -y_1 - 3y_2 + y_3, y_1 - y_3 - 3y_3\big)$$

and

$$\mathrm{v} = -(y_2, y_3, y_1).$$

Note that if $\mathrm{v} = (v_1, v_2, v_3)$, then y can be obtained by

$$\mathrm{y} = -\mathrm{R}\mathrm{v} = -(v_3, v_1, v_2).$$

Three lines

Assume we are given three lines

$$C_1 = c_1 + \mathbb{R}u_1,\ C_2 = c_2 + \mathbb{R}u_2,\ C_3 = c_3 + \mathbb{R}u_3,$$

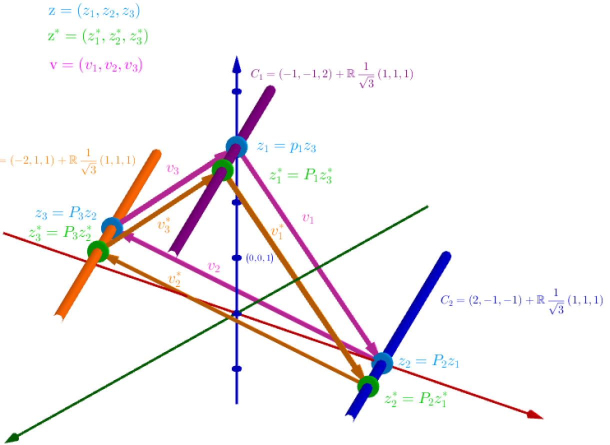
where

$$c_1 \perp u_1,\ c_2 \perp u_2,\ c_3 \perp u_3 \text{ and } \|u_1\| = \|u_2\| = \|u_3\| = 1.$$

The parallel case: WLOG $u_1 = u_2 = u_3 = u$. Then

$$\mathsf{Z} = (c_1, c_2, c_3) + \mathbb{R}(u, u, u) \text{ and } \mathsf{v} = (c_2 - c_1, c_3 - c_2, c_1 - c_3).$$

Three parallel lines



Three nonparallel lines

The nonparallel case: Then $\mathsf{Z} = \{\mathsf{z}\}$ is a singleton and

$$\mathsf{z} = (z_1, z_2, z_3) = (c_1 + \rho_1 u_1, c_2 + \rho_2 u_2, c_3 + \rho_3 u_3),$$

where

$$\rho_1 := \frac{\langle u_1, c_3\rangle + \langle u_1, u_3\rangle\langle u_3, c_2\rangle + \langle u_1, u_3\rangle\langle u_3, u_2\rangle\langle u_2, c_1\rangle}{1 - \langle u_3, u_2\rangle\langle u_2, u_1\rangle\langle u_1, u_3\rangle},$$

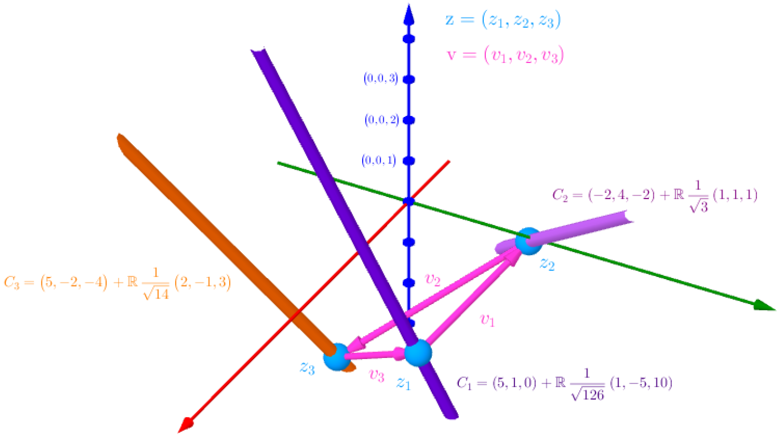
$$\rho_2 := \frac{\langle u_2, c_1\rangle + \langle u_2, u_1\rangle\langle u_1, c_3\rangle + \langle u_2, u_1\rangle\langle u_1, u_3\rangle\langle u_3, c_2\rangle}{1 - \langle u_3, u_2\rangle\langle u_2, u_1\rangle\langle u_1, u_3\rangle},$$

$$\rho_3 := \frac{\langle u_3, c_2\rangle + \langle u_3, u_2\rangle\langle u_2, c_1\rangle + \langle u_3, u_2\rangle\langle u_2, u_1\rangle\langle u_1, c_3\rangle}{1 - \langle u_3, u_2\rangle\langle u_2, u_1\rangle\langle u_1, u_3\rangle},$$

and

$$\mathsf{v} = (z_2 - z_1, z_3 - z_2, z_1 - z_3).$$

Three nonparallel lines



An example featuring epi(exp)

Let's assume that

$$X = \mathbb{R}^2$$

and consider the following three sets under two different orderings:

$$\mathrm{epi}(\exp) = \{(\xi, \eta) \in \mathbb{R}^2 \mid \exp(\xi) \leq \eta\},$$

$$\mathbb{R} \times \{0\},$$

and

$$\mathbb{R} \times \{1\}.$$

(This example is similar to one discussed by De Pierro in 2001.)

An ordering with cycles

Assume first that

$$C_1 = \mathbb{R} \times \{0\}, \quad C_2 = \mathbb{R} \times \{1\}, \quad C_3 = \mathrm{epi}(\exp).$$

Then, using the characterizations, one can verify that

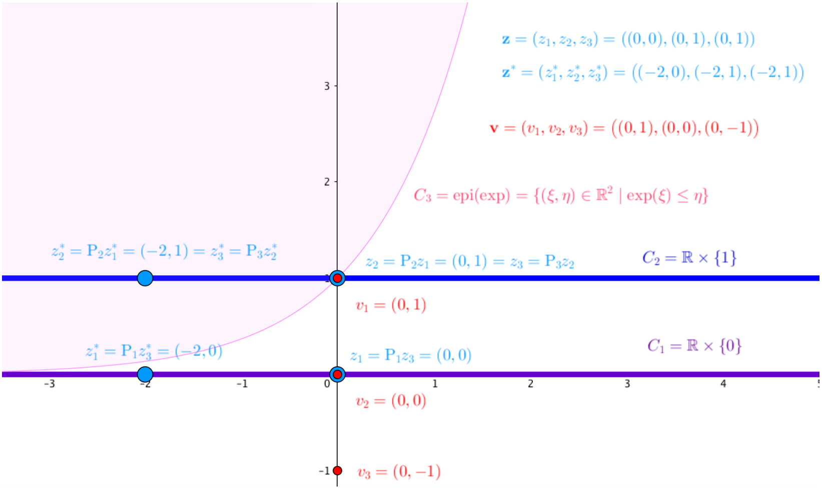
$$\begin{aligned}
\mathrm{y} &= (y_1, y_2, y_3) = \big((0,1),(0,-1),(0,0)\big), \\
\mathrm{e} &= (e_1, e_2, e_3) = \big((0,-\tfrac{2}{3}),(0,\tfrac{1}{3}),(0,\tfrac{1}{3})\big), \\
\mathrm{v} &= (v_1, v_2, v_3) = \big((0,1),(0,0),(0,-1)\big).
\end{aligned}$$

and that

$$\begin{aligned}
F_3 &= C_3 \cap (C_2 + v_2) \cap (C_1 + v_1 + v_2) \\
&= \mathrm{epi}(\exp) \cap (\mathbb{R} \times \{1\} + (0,0)) \cap (\mathbb{R} \times \{0\} + (0,1)) \\
&= \mathrm{epi}(\exp) \cap (\mathbb{R} \times \{1\}) \cap (\mathbb{R} \times \{1\}) \\
&= \mathbb{R}_- \times \{1\}.
\end{aligned}$$

Similarly, $F_1 = \mathbb{R}_- \times \{0\}$ and $F_2 = \mathbb{R}_- \times \{1\}$.

The ordering with cycles



An ordering without cycles

Now assume that

$$C_1 = \mathbb{R} \times \{1\}, \quad C_2 = \mathbb{R} \times \{0\}, \quad C_3 = \mathrm{epi}(\exp).$$

Then, using the characterizations, one can verify that

$$\begin{aligned}
\mathrm{y} &= (y_1, y_2, y_3) = \big((0,-1),(0,1),(0,0)\big), \\
\mathrm{e} &= (e_1, e_2, e_3) = \big((0,\tfrac{2}{3}),(0,-\tfrac{1}{3}),(0,-\tfrac{1}{3})\big), \\
\mathrm{v} &= (v_1, v_2, v_3) = \big((0,-1),(0,0),(0,1)\big).
\end{aligned}$$

Hence

$$\begin{aligned}
F_3 &= C_3 \cap (C_2 + v_2) \cap (C_1 + v_1 + v_2) \\
&= \mathrm{epi}(\exp) \cap (\mathbb{R} \times \{0\} + (0,0)) \cap (\mathbb{R} \times \{1\} + (0,-1)) \\
&= \mathrm{epi}(\exp) \cap (\mathbb{R} \times \{1\}) \cap (\mathbb{R} \times \{0\}) \\
&= \varnothing,
\end{aligned}$$

and therefore $F_1 = F_2 = \varnothing$ as well.

Computing the magic vector y

A characterization of y

One may show that y is the unique fixed point of the operator

$$(\mathrm{Id} - \mathrm{P})(\tfrac{1}{2}\mathrm{y} - \mathrm{Ty}) = \mathrm{y}, \quad \text{where } \mathrm{P} = \mathrm{P}_{\overline{\mathrm{C}+\Delta}}.$$

Now P is a projector, hence $\mathrm{Id} - \mathrm{P}$ is (firmly) nonexpansive.
Is $\frac{1}{2}\,\mathrm{Id} - \mathrm{T}$ a Banach contraction? It depends on m!

m	eigenvalues of $(\frac{1}{2}\,\mathrm{Id} - \mathrm{T})^*(\frac{1}{2}\,\mathrm{Id} - \mathrm{T})$	$\|\frac{1}{2}\,\mathrm{Id} - \mathrm{T}\|$
2	$\frac{1}{4}$ (twice)	$\frac{1}{2} = 0.5$
3	$\frac{1}{3}$ (twice), $\frac{1}{4}$	$\frac{1}{\sqrt{3}} \approx 0.58$
4	$\frac{1}{2}$ (twice), $\frac{1}{4}$ (twice)	$\frac{1}{\sqrt{2}} \approx 0.71$
5	$\frac{1}{2} + \frac{1}{2\sqrt{5}}$ (twice), $\frac{1}{2} - \frac{1}{2\sqrt{5}}$ (twice), $\frac{1}{4}$	$\sqrt{\frac{1}{2} + \frac{1}{2\sqrt{5}}} \approx 0.85$
6	1 (twice), $\frac{1}{3}$ (twice), $\frac{1}{4}$ (twice)	1

If $m \geq 7$, then $\|\frac{1}{2}\,\mathrm{Id} - \mathrm{T}\| > 1$.

A forward-backward approach

One may also show the following, using the forward-backward algorithm appropriately:

Let $0 < \gamma < 1$, $x_0 \in X$, generate a sequence $(x_n)_{n\in\mathbb{N}}$ via

$$\begin{aligned} x_{n+1} &= P_{\overline{C+\Delta}}\big(x_n - \gamma(\tfrac{1}{2}\,\mathrm{Id} + T)^{-1}x_n\big) \\ &= P_{\overline{C+\Delta}}\big((1-\gamma)x_n + \gamma R x_n - 2\gamma P_\Delta x_n\big). \end{aligned}$$

Then

$$x_n \to e,$$

$$R x_n - x_n - 2P_\Delta x_n \to y,$$

and

$$R^* x_n - x_n \to v.$$

But how to compute $P_{\overline{C+\Delta}}$?

Seeger's algorithm

Given

$$x \in X \text{ and } d_0 \in X,$$

generate sequences $(c_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ iteratively via

$$c_n := P_C(x - d_{n-1}), \quad d_n := P_\Delta(x - c_n).$$

Then

$$c_n + d_n \to P_{\overline{C+\Delta}}(x).$$

This was proved by A. Seeger: Alternating projection and decomposition with respect to two convex sets, *Mathematica Japonica* 47 (1998), 273–280.

Conclusion

Conclusion and future work future work

- The geometry conjecture is settled.
- The proof relied on monotone operator theory!
- We have presented new descriptions of the cycles and the difference vectors.

The following questions are natural to investigate:

- How do these results generalize from projections to underrelaxed projections or even proximal mappings? (Partial results are available!)
- Can we even go to maximally monotone operators, perhaps utilizing the extended or variational sum?
- Can we rigorously justify algorithms that do not rely on Seeger's algorithm?

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