Acceleration of first-order algorithms via inertial dynamics with Hessian damping

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Introduction

Convex optimization

- $\mathcal{H}$ real Hilbert space, $\langle x, x\rangle = \|x\|^2$.
- $f : \mathcal{H} \to \mathbb{R}$ convex differentiable, $S = \operatorname{argmin}_{\mathcal{H}} f \neq \emptyset$.

$$(\mathcal{P}) \quad \min\left\{f(x) :\ x \in \mathcal{H}\right\}.$$

Damped inertial dynamic

$$\ddot{x}(t) + \underbrace{\frac{\alpha}{t}\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t)}_{\text{damping force}} + \underbrace{b(t)\nabla f(x(t))}_{\text{driving force}} = 0.$$

$$\text{Damping:} \left\{\begin{array}{ll} \dfrac{\alpha}{t}\dot{x}(t) : & \text{accelerated gradient method of Nesterov;} \\ \beta\nabla^2 f(x(t))\dot{x}(t) : & \text{neutralization of oscillations.} \end{array}\right.$$

Temporal discretization, step size $\sqrt{s}$

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + b(t)\nabla f(x(t)) = 0.$$

$$\nabla^2 f(x(t))\dot{x}(t) = \tfrac{d}{dt}\nabla f(x(t)) \longrightarrow \text{ first-order algorithms.}$$

$$\begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) - \beta\sqrt{s}\left(\nabla f(x_k) - \nabla f(x_{k-1})\right) - \frac{\beta\sqrt{s}}{k}\nabla f(x_{k-1}) \\ x_{k+1} = y_k - s\nabla f(y_k). \end{cases}$$

Convergence rates

$f : \mathcal{H} \to \mathbb{R}$ convex, ∇f L-Lipschitz, $\alpha \geq 3$, $\quad 0 < \beta < 2\sqrt{s}$, $\quad sL \leq 1$.

$$i)\ f(x_k) - \min_{\mathcal{H}} f = \mathcal{O}\left(\frac{1}{k^2}\right) \text{ as } k \to +\infty;$$

$$ii)\quad \sum_k k^2\|\nabla f(x_k)\|^2 < +\infty \text{ and } \sum_k k^2\|\nabla f(y_k)\|^2 < +\infty.$$

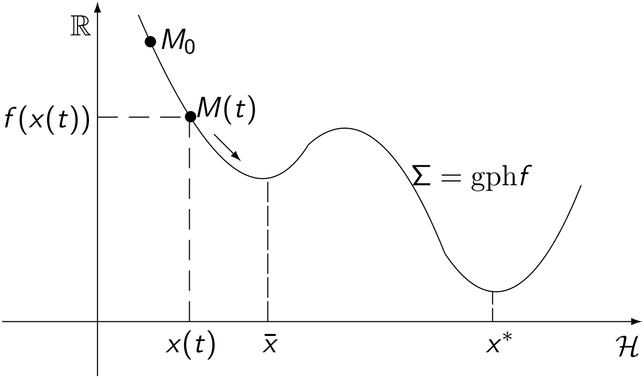
Contents

1. INERTIAL DYNAMICS/ALGORITHMS IN OPTIMIZATION

1. The heavy ball with friction method of Polyak (64,87)

$\gamma > 0$: fixed viscous damping coefficient

$$\text{(HBF)} \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla f(x(t)) = 0.$$



Exploration of local minima via (HBF)

The heavy ball with friction method of Polyak

$$\text{(HBF)} \qquad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla f(x(t)) = 0.$$

General convex case

- $f(x(t)) - \inf_{\mathcal{H}} f = \mathcal{O}\left(\frac{1}{t}\right)$ as $t \to +\infty$.
- $x(t) \rightharpoonup x_\infty \in S$ weakly (Alvarez, 2000).

Strongly convex case: $f - \frac{\mu}{2}\|\cdot\|^2$ convex

$$\ddot{x}(t) + 2\sqrt{\mu}\dot{x}(t) + \nabla f(x(t)) = 0.$$

- $f(x(t)) - \inf_{\mathcal{H}} f = \mathcal{O}\left(e^{-\sqrt{\mu}t}\right)$ as $t \to +\infty$.
- Link between the geometry of f and the damping coefficient.

Recent trends in dissipative autonomous systems:

Haraux-Jendoubi (Springer Briefs, 2015), A.-Bot-Csetnek (JEMS, 2021).

2. Asymptotic vanishing damping: $\gamma(t) \to 0$ as $t \to +\infty$

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla f(x(t)) = 0, \quad t \geq t_0.$$

Theorem (Cabot-Engler-Gaddat (TAMS 2009))

The optimization property is satisfied if $\int_{t_0}^{+\infty} \gamma(t)dt = +\infty.$

Define: $p(t) := \exp\left(\int_{t_0}^{t} \gamma(\tau)d\tau\right)$, $\Gamma_\gamma(t) := p(t)\int_t^{+\infty} \frac{ds}{p(s)}$.

Theorem (A.-Cabot (JDE 2017))

$$\gamma(t)\Gamma_\gamma(t) \leq m \quad \text{for some } m < \tfrac{3}{2}.$$

$$\Downarrow$$

$$f(x(t)) - \min_{\mathcal{H}} f = \left(\frac{1}{\int_{t_0}^{t} \Gamma_\gamma(s)\, ds}\right), \quad \|\dot{x}(t)\| = o\left(\frac{1}{\int_{t_0}^{t} \Gamma_\gamma(s)\, ds}\right)^{1/2}.$$

Case: $\gamma(t) = \frac{\alpha}{t}$, $\Gamma_\gamma(t) = \frac{t}{\alpha - 1}$, which gives $\alpha > 3$.

Su-Boyd-Candès model for Nesterov accelerated method

$$(\mathrm{AVD})_\alpha \qquad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla f(x(t)) = 0.$$

- $\alpha = 3$: Su-Boyd-Candès (NIPS 2014)

$$f(x(t)) - \min_{\mathcal{H}} f = \mathcal{O}\left(\frac{1}{t^2}\right) \text{ as } t \to +\infty.$$

- $\alpha > 3$: A.-Chbani-Peypouquet-Redont (Math. Program. 2018)

$$f(x(t)) - \min_{\mathcal{H}} f = o\left(\frac{1}{t^2}\right), \; x(t) \rightharpoonup x_\infty \in S \text{ as } t \to +\infty.$$

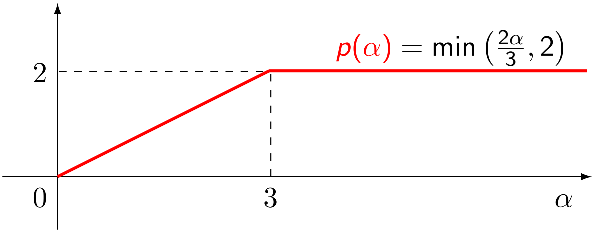
- $\alpha \leq 3$: Apidopoulos-Aujol-Dossal (SIOPT 2018),
  A.-Chbani-Riahi (ESAIM COCV 2019)

$$f(x(t)) - \min_{\mathcal{H}} f = \mathcal{O}\left(\frac{1}{t^{\frac{2\alpha}{3}}}\right) \text{ as } t \to +\infty.$$

$$(\mathrm{AVD})_\alpha \qquad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla f(x(t)) = 0.$$

$$f(x(t)) - \min_{\mathcal{H}} f = \mathcal{O}\left(\frac{1}{t^{p(\alpha)}}\right) \text{ as } t \to +\infty.$$

Optimal rate: $\begin{cases} \alpha \geq 3: \ f(x) = \|x\|^r, r \to +\infty; \\ \alpha < 3: \ f(x) = \|x\|. \end{cases}$



$$(\text{AVD})_\alpha \qquad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla f(x(t)) = 0.$$

Temporal discretization, h step size, $s = h^2$

$$\frac{1}{s}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha}{ks}(x_k - x_{k-1}) + \nabla f(y_k) = 0.$$

Different choices for y_k

- Explicit: $y_k = x_k$, Heavy Ball with Friction, Polyak

$$(\text{HBF}) \; x_{k+1} = x_k + \left(1 - \tfrac{\alpha}{k}\right)(x_k - x_{k-1}) - s\nabla f(x_k).$$

- Implicit: $y_k = x_{k+1}$, Inertial Proximal algorithm, Güler, Beck-Teboulle

$$(\text{IP})_\alpha \qquad x_{k+1} = \text{prox}_{sf}\left(x_k + \left(1 - \tfrac{\alpha}{k}\right)(x_k - x_{k-1})\right).$$

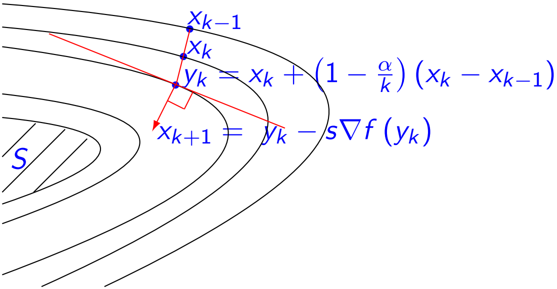
- Nesterov: $\quad y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1})$ Inertial Gradient algorithm

$$(\text{IG})_\alpha \quad \begin{cases} y_k = x_k + \left(1 - \tfrac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} = y_k - s\nabla f(y_k). \end{cases}$$

$$\min \{f(x) : \ x \in \mathcal{H}\}, \ f : \mathcal{H} \to \mathbb{R} \text{ convex differentiable}, \ S = \operatorname{argmin} f \neq \emptyset.$$

Inertial Gradient algorithm, Nesterov (1983, 2004)

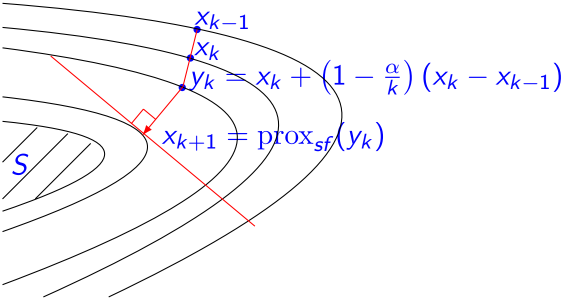
$$(\mathrm{IG})_\alpha \quad \begin{cases} y_k &= x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} &= y_k - s\nabla f(y_k) \end{cases}$$



$\min\{f(x):\ x \in \mathcal{H}\},\ f:\mathcal{H} \to \mathbb{R}\cup\{+\infty\}$ closed, convex, $S = \mathrm{argmin} f \neq \emptyset$.

Inertial Proximal algorithm, $\mathrm{prox}_{sf}(y) := \mathrm{argmin}_{\xi\in\mathcal{H}}\{f(\xi) + \frac{1}{2s}\|y-\xi\|^2\}$

$$(\mathrm{IP})_\alpha \quad \begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} = \mathrm{prox}_{sf}(y_k). \end{cases}$$



- $f : \mathcal{H} \to \mathbb{R}$ convex, $\mathcal{C}^1$, ∇f L-Lipschitz continuous; $0 < s \leq \frac{1}{L}$.
- $g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ convex, lower semicontinuous, proper.

$$
(\mathrm{IPG})_\alpha \quad \left\{ \begin{array}{rcl} y_k & = & x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} & = & \mathrm{prox}_{sg}\left(y_k - s\nabla f(y_k)\right) \end{array} \right.
$$

- $\alpha = 3$: $(f+g)(x_k) - \min_{\mathcal{H}}(f+g) = \mathcal{O}(\frac{1}{k^2})$,

  Nesterov (1983), Beck-Teboulle: FISTA (SIAM J. Imaging 2009).

- $\alpha > 3$: $(f+g)(x_k) - \min_{\mathcal{H}}(f+g) = o(\frac{1}{k^2})$, $x_k \rightharpoonup x_\infty \in S$,

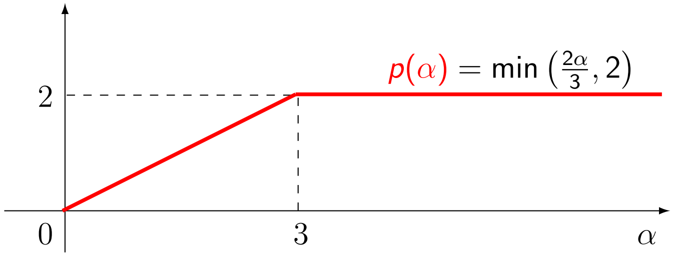
  Chambolle-Dossal (JOTA 2015), A.-Peypouquet (SIOPT 2016).

- $\alpha \leq 3$: $(f+g)(x_k) - \min_{\mathcal{H}}(f+g) = \mathcal{O}\left(\frac{1}{k^{\frac{2\alpha}{3}}}\right)$.

  Apidopoulos-Aujol-Dossal (Math Prog '20), A.-Chbani-Riahi (COCV '18)

Convergence rate of the Inertial Proximal Gradient algorithm

$$(\mathrm{IPG})_\alpha \quad \begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} = \operatorname{prox}_{sg}\left(y_k - s\nabla f(y_k)\right) \end{cases}$$



$$(f+g)(x_k) - \min_{\mathcal{H}}(f+g) = \mathcal{O}\left(\frac{1}{k^{p(\alpha)}}\right) \text{ as } k \to +\infty.$$

2. INERTIAL DYNAMICS WITH HESSIAN DRIVEN DAMPING

Historical aspects

- (HBF) $\ddot{x}(t) + \Gamma \dot{x}(t) + \nabla f(x(t)) = 0.$

  $\Gamma : \mathcal{H} \to \mathcal{H}$ anisotropic, Alvarez (SICON 2000).

- $(\text{DIN})_\beta$ $\ddot{x}(t) + \gamma \dot{x}(t) + \beta \nabla^2 f(x(t)) \dot{x}(t) + \nabla f(x(t)) = 0.$

  Alvarez-A.-Bolte-Redont (JMPA 2002), A.-Maingé-Redont (DEA 2012).

- $(\text{DIN}-\text{AVD})_{\alpha,\beta}$ $\ddot{x}(t) + \frac{\alpha}{t} \dot{x}(t) + \beta \nabla^2 f(x(t)) \dot{x}(t) + \nabla f(x(t)) = 0.$

  A.-Peypouquet-Redont (JDE '16), A.-Chbani-Fadili-Riahi (Math Prog '20)

Related works

Shi-Du-Jordan-Su (arXiv:1810.08907, '18), Lin-Jordan (arXiv:1912.07168, '19)

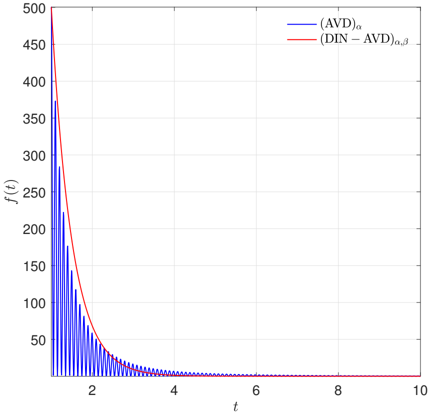
Castera-Bolte-Févotte-Pauwels (hal-02140748, 2019),

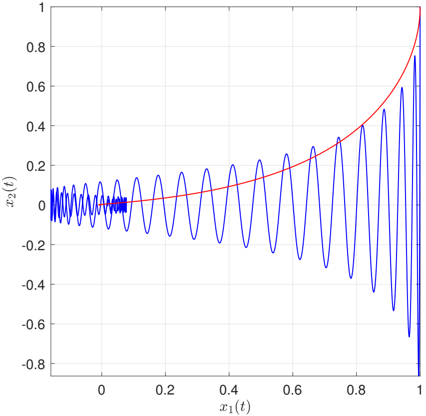
Bot-Csetnek-László (Math. Program., 2019),

Alecsa-László-Pinta (AMO '20), Adly-A. (SIOPT 2020).

Compare $(\mathrm{AVD})_\alpha$ with $(\mathrm{DIN}-\mathrm{AVD})_{\alpha,\beta}$

- $f(x_1, x_2) = \frac{1}{2}(x_1^2 + 1000x_2^2)$: ill-conditioned.
- $\alpha = 3.1$, $\beta = 1$.
- Initial conditions: $(x_1(1), x_2(1)) = (1, 1)$, $(\dot{x}_1(1), \dot{x}_2(1)) = (0, 0)$.





Link with the Newton method

- f convex, $\mathcal{C}^2$, solve $\nabla f(x) = 0$ by the Newton method.
- $\nabla f(x_k) + \nabla^2 f(x_k)(x_{k+1} - x_k) = 0$.
- Continuous version: $\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0$: Ill-posed.
- Levenberg-Marquardt regularization, $\gamma(t) > 0$

$$\gamma(t)\dot{x}(t) + \nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0.$$

Well-posed: A.-Svaiter (SICON, 2011). Valid with a general maximally monotone operator, closed loop form.

- Dynamical Inertial Newton method

$$\text{(DIN)} \quad \ddot{x}(t) + \gamma(t)\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0.$$

$$(\mathrm{DIN}-\mathrm{AVD})_{\alpha,\beta} \quad \ddot{x}(t)+\frac{\alpha}{t}\dot{x}(t)+\beta\nabla^2 f(x(t))\dot{x}(t)+\nabla f(x(t))=0.$$

Lyapunov analysis: A.-Peypouquet-Redont (JDE 2016) (APR for short)

Theorem (APR)

Let $x:[t_0,+\infty[\to\mathcal{H}$ *be a solution trajectory of* $(\mathrm{DIN}-\mathrm{AVD})_{\alpha,\beta}$. *Suppose* $\alpha\geq 3$, $\beta\geq 0$. *Then, as* $t\to+\infty$

$$f(x(t))-\min_{\mathcal{H}} f=\mathcal{O}\left(\frac{1}{t^2}\right).$$

In addition, when $\beta>0$: $\displaystyle\int_{t_0}^{+\infty} t^2\|\nabla f(x(t))\|^2 dt<+\infty.$

Lyapunov function: $x^*\in\operatorname{argmin}_{\mathcal{H}} f$, $m=\min_{\mathcal{H}} f=f(x^*)$

$$\mathcal{E}_{\alpha,\beta}(t):=t(t-\beta)\left(f(x(t))-m\right)+\tfrac{1}{2}\|(\alpha-1)(x(t)-x^*)+t\left(\dot{x}(t)+\beta\nabla f(x(t))\right)\|^2.$$

$$(\mathrm{DIN-AVD})_{\alpha,\beta} \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0.$$

Lyapunov analysis

$$\mathcal{E}_{\alpha,\beta}(t) := t(t-\beta)\left(f(x(t)) - m\right) + \frac{1}{2}\|(\alpha-1)(x(t)-x^*) + t\left(\dot{x}(t) + \beta\nabla f(x(t))\right)\|^2.$$

Derivation of $\mathcal{E}_{\alpha,\beta}(\cdot)$

$$\dot{\mathcal{E}}_{\alpha,\beta}(t) + \Big((\alpha-3)t - \beta(\alpha-2)\Big)\Big(f(x(t)) - f(x^*)\Big) + \beta t(t-\beta)\|\nabla f(x(t))\|^2 \leq 0.$$

- $\alpha > 3$, $\quad t \geq t_1 := \beta\frac{\alpha-2}{\alpha-3} \implies \dot{\mathcal{E}}_{\alpha,\beta}(t) \leq 0 \quad i.e. \quad \mathcal{E}_{\alpha,\beta}(\cdot)$ decreasing.

$\forall t \geq t_1 \quad \mathcal{E}_{\alpha,\beta}(t) \leq \mathcal{E}_{\alpha,\beta}(t_1) \implies f(x(t)) - \min_{\mathcal{H}} f \leq \frac{\mathcal{E}_{\alpha,\beta}(t_1)}{t(t-\beta)} = \mathcal{O}\left(\frac{1}{t^2}\right).$

- $\beta > 0$, integration $\implies \displaystyle\int_{t_0}^{\infty} t^2\|\nabla f(x(t))\|^2 dt < +\infty.$

$$(\mathrm{DIN}-\mathrm{AVD})_{\alpha,\beta} \quad \ddot{x}(t)+\frac{\alpha}{t}\dot{x}(t)+\beta\nabla^2 f(x(t))\dot{x}(t)+\nabla f(x(t))=0.$$

Theorem (APR)

Let $x:[t_0,+\infty[\to\mathcal{H}$ *be a solution trajectory of* $(\mathrm{DIN}-\mathrm{AVD})_{\alpha,\beta}$. *Suppose that* $\alpha>3$, $\beta\geq 0$. *Then, as* $t\to+\infty$

i) $x(t)\rightharpoonup x_\infty\in\operatorname{argmin}_{\mathcal{H}} f$

ii) $f(x(t))-\min_{\mathcal{H}} f=o\left(\frac{1}{t^2}\right), \quad \|\dot{x}(t)+\beta\nabla f(x(t))\|=o\left(\frac{1}{t}\right).$

iii) $\int_{t_0}^{+\infty} t\left(f(x(t))-\min_{\mathcal{H}} f\right)dt<+\infty, \quad \int_{t_0}^{+\infty} t\|\dot{x}(t)\|^2 dt<+\infty.$

Proof: Lyapunov analysis via the anchoring functions

$h_z(t)=\frac{1}{2}\|x(t)-z\|^2$ $(z\in\operatorname{argmin}_{\mathcal{H}} f)$, and Opial's lemma.

Perturbed dynamic, $e : [t_0, +\infty[\to \mathcal{H}$

$$(\mathrm{DIN-AVD})_{\mathrm{pert}} \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = e(t).$$

Theorem (APR)

Let $x : [t_0, +\infty[\to \mathcal{H}$ *be a solution trajectory of* $(\mathrm{DIN-AVD})_{\mathrm{pert}}$.

Suppose that $\displaystyle\int_{t_0}^{+\infty} t\,\|e(t)\|\,dt < +\infty$. *Then,*

- $\alpha \geq 3$, $\beta \geq 0$: $\quad f(x(t)) - \min_{\mathcal{H}} f = \mathcal{O}\left(\dfrac{1}{t^2}\right)$ *as* $t \to +\infty$.
- $\alpha > 3$, $\beta \geq 0$: $\quad x(t) \rightharpoonup x_\infty \in \mathrm{argmin}_{\mathcal{H}} f$ *as* $t \to +\infty$.
- $\alpha \geq 3$, $\beta > 0$: $\quad \displaystyle\int_{t_0}^{+\infty} t^2\|\nabla f(x(t))\|^2 dt < +\infty$.

Strongly convex case

Theorem (ACFR)

Suppose that $f : \mathcal{H} \to \mathbb{R}$ *is* μ*-strongly convex for some* $\mu > 0$.
Let $x(\cdot) : [t_0, +\infty[\to \mathcal{H}$ *be a solution trajectory of*

$$\ddot{x}(t) + 2\sqrt{\mu}\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0.$$

Suppose that $0 \le \beta \le \frac{1}{2\sqrt{\mu}}$. *Then, for all* $t \ge t_0$

$$i) \quad \frac{\mu}{2}\|x(t) - x^\star\|^2 \le f(x(t)) - \min_{\mathcal{H}} f \le Ce^{-\frac{\sqrt{\mu}}{2}(t-t_0)},$$

where $C := f(x(t_0)) - \min_{\mathcal{H}} f + \mu \mathrm{dist}(x(t_0), S)^2 + \|\dot{x}(t_0) + \beta\nabla f(x(t_0))\|^2$.

$$ii) \quad e^{-\sqrt{\mu}t}\int_{t_0}^{t} e^{\sqrt{\mu}s}\|\nabla f(x(s))\|^2 ds \le C_1 e^{-\frac{\sqrt{\mu}}{2}t}.$$

Moreover, $\int_{t_0}^{\infty} e^{\frac{\sqrt{\mu}}{2}t}\|\dot{x}(t)\|^2 dt < +\infty$.

3. INERTIAL ALGORITHMS
with
HESSIAN DRIVEN DAMPING

A.-Chbani-Fadili-Riahi (Math. Program. 2020), (ACFR) for short.

$f : \mathcal{H} \to \mathbb{R}$ convex, ∇f L-Lipschitz continuous.

Temporal rescaling of $(\text{DIN-AVD})_{\alpha,\beta}$

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + \left(1 + \frac{\beta}{t}\right)\nabla f(x(t)) = 0.$$

Temporal discretization: $s = h^2$, $\nabla^2 f(x(t))\dot{x}(t) = \frac{d}{dt}\nabla f(x(t))$.

$$\frac{1}{s}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha}{ks}(x_k - x_{k-1}) + \frac{\beta}{\sqrt{s}}(\nabla f(x_k) - \nabla f(x_{k-1}))$$

$$+\frac{\beta}{k\sqrt{s}}\nabla f(x_{k-1}) + \nabla f(y_k) = 0.$$

$$\frac{1}{s}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha}{ks}(x_k - x_{k-1}) + \frac{\beta}{\sqrt{s}}(\nabla f(x_k) - \nabla f(x_{k-1})) + \frac{\beta}{k\sqrt{s}}\nabla f(x_{k-1}) + \nabla f(y_k) = 0.$$

Choose $y_k \approx$ Nesterov's accelerated gradient method, set $\alpha_k = 1 - \frac{\alpha}{k}$.

(IGAHD): Inertial Gradient Algorithm with Hessian Damping

$$\begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}) - \beta\sqrt{s}\,(\nabla f(x_k) - \nabla f(x_{k-1})) - \frac{\beta\sqrt{s}}{k}\nabla f(x_{k-1}) \\ x_{k+1} = y_k - s\nabla f(y_k). \end{cases}$$

Related algorithm: Shi-Du-Jordan-Su (arXiv 2018).

Lyapunov analysis, $x^\star \in \operatorname{argmin}_{\mathcal{H}} f$, $\quad t_k := \frac{k-1}{\alpha-1}$.

$$\mathcal{E}_k := t_k^2(f(x_k) - f(x^\star)) + \frac{1}{2s}\|v_k\|^2$$

$$v_k := (x_{k-1} - x^\star) + t_k\Big(x_k - x_{k-1} + \beta\sqrt{s}\nabla f(x_{k-1})\Big).$$

Theorem (ACFR, 2019)

- $f : \mathcal{H} \to \mathbb{R}$ *convex*, ∇f L*-Lipschitz continuous*, $\operatorname{argmin}_{\mathcal{H}} f \neq \emptyset$.
- $\alpha \geq 3$, $0 < \beta < 2\sqrt{s}$, $sL \leq 1$.

$(x_k)_{k\in\mathbb{N}}$ *generated by* (IGAHD). *Then* $(\mathcal{E}_k)_{k\in\mathbb{N}}$ *is non-increasing and*

$$i)\ f(x_k) - \min_{\mathcal{H}} f = \mathcal{O}\left(\frac{1}{k^2}\right) \ \textit{as}\ k \to +\infty;$$

$$ii) \sum_k k^2\|\nabla f(y_k)\|^2 < +\infty \ \textit{and} \ \sum_k k^2\|\nabla f(x_k)\|^2 < +\infty.$$

$$iii)\ \textit{If}\ \alpha > 3, \textit{then}\ (x_k)\ \textit{converges weakly to some}\ x^\star \in \operatorname{argmin}_{\mathcal{H}} f.$$

Reinforced version of the gradient descent lemma.
Since $s \leq \frac{1}{L}$, f convex, and ∇f is L-lipschitz continuous,

$$f(y - s\nabla f(y)) \leq f(x) + \langle \nabla f(y), y - x\rangle - \frac{s}{2}\|\nabla f(y)\|^2 - \frac{s}{2}\|\nabla f(x) - \nabla f(y)\|^2.$$

Write it successively at $y = y_k$ and $x = x_k$, then at $y = y_k$, $x = x^\star$.

$$f(x_{k+1}) \leq f(x_k) + \langle \nabla f(y_k), y_k - x_k\rangle - \frac{s}{2}\|\nabla f(y_k)\|^2 - \frac{s}{2}\|\nabla f(x_k) - \nabla f(y_k)\|^2$$
$$f(x_{k+1}) \leq f(x^\star) + \langle \nabla f(y_k), y_k - x^\star\rangle - \frac{s}{2}\|\nabla f(y_k)\|^2 - \frac{s}{2}\|\nabla f(y_k)\|^2.$$

Linear combination of the two above equations gives

$$\begin{aligned} t_{k+1}^2(f(x_{k+1}) - f(x^\star)) &\leq (t_{k+1}^2 - t_{k+1} - t_k^2)(f(x_k) - f(x^\star)) + t_k^2(f(x_k) - f(x^\star)) \\ &+ t_{k+1}\langle \nabla f(y_k), (t_{k+1} - 1)(y_k - x_k) + y_k - x^\star\rangle - \frac{s}{2}t_{k+1}^2\|\nabla f(y_k)\|^2 \\ &- \frac{s}{2}(t_{k+1}^2 - t_{k+1})\|\nabla f(x_k) - \nabla f(y_k)\|^2 - \frac{s}{2}t_{k+1}\|\nabla f(y_k)\|^2. \end{aligned}$$

Since $\alpha \geq 3$ we have $t_{k+1}^2 - t_{k+1} - t_k^2 \leq 0$...

Strongly convex case

$f : \mathcal{H} \to R$ μ-strongly convex, ∇f is L-Lipschitz continuous.

$$\ddot{x}(t) + 2\sqrt{\mu}\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0.$$

Explicit time discretization with centered finite differences

$$\frac{1}{s}(x_{k+1}-2x_k+x_{k-1})+\frac{\sqrt{\mu}}{\sqrt{s}}(x_{k+1}-x_{k-1})+\beta\frac{1}{\sqrt{s}}(\nabla f(x_k)-\nabla f(x_{k-1}))+\nabla f(x_k) = 0.$$

$$x_{k+1} = x_k + \frac{1-\sqrt{\mu s}}{1+\sqrt{\mu s}}(x_k - x_{k-1}) - \frac{\beta\sqrt{s}}{1+\sqrt{\mu s}}\left(\nabla f(x_k) - \nabla f(x_{k-1})\right) - \frac{s}{1+\sqrt{\mu s}}\nabla f(x_k).$$

Theorem (ACFR)

$\beta \leq \frac{1}{\sqrt{\mu}},\ L \leq \min\left\{\frac{\sqrt{\mu}}{8\beta}, \frac{\frac{\sqrt{\mu}}{2s}+\frac{\mu}{\sqrt{s}}}{2\beta\mu+\frac{1}{\sqrt{s}}+\frac{\sqrt{\mu}}{2}}\right\}$. *Set* $q = \frac{1}{1+\frac{1}{2}\sqrt{\mu s}}$, $\theta = \frac{1}{1+\sqrt{\mu s}}$.

(i) $\|x_k - x^\star\| = \mathcal{O}\left(q^{k/2}\right)$ *and* $f(x_k) - \min_{\mathcal{H}} f = \mathcal{O}\left(q^k\right)$ *as* $k \to +\infty$.

(ii) $\theta^k \sum_{p=0}^{k-2} \theta^{-j}\|\nabla f(x_j)\|^2 = \mathcal{O}\left(q^k\right)$ *as* $k \to +\infty$.

Nonsmooth convex case

$f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ μ-strongly convex.
Idea: replace f with its Moreau envelope.
Preserves the infimal value and the solution set.

Proximal calculus (Bauschke-Combettes)

- f μ-strongly convex $\Longrightarrow$ f_λ strongly convex with modulus $\frac{\mu}{1+\lambda\mu}$.
- $\nabla f_\lambda(x) = \frac{1}{\lambda}\left(x - \mathrm{prox}_{\lambda f}(x)\right)$,
- $\mathrm{prox}_{\theta f_\lambda}(x) = \frac{\lambda}{\lambda+\theta} x + \frac{\theta}{\lambda+\theta} \mathrm{prox}_{(\lambda+\theta) f}(x)$.

$$
\begin{cases}
y_k = x_k + (1-a)(x_k - x_{k-1}) + \frac{\beta\sqrt{s}}{\lambda}(1-a)\left(x_k - \mathrm{prox}_{\lambda f}(x_k)\right) \\
x_{k+1} = \frac{\lambda}{\lambda+\theta} y_k + \frac{\theta}{\lambda+\theta} \mathrm{prox}_{(\lambda+\theta) f}(y_k).
\end{cases}
$$

Similar type of convergence rates.

4. NUMERICAL EXPERIMENTS

Regularized Least Square (signal/image, machine learning, statistics)

$$\text{(RLS)} \quad \min_{x \in \mathbb{R}^n} \left\{ f(x) := \frac{1}{2}\|Ax - b\|^2 + g(x) \right\}$$

- A linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$, $m \leq n$, $b \in \mathbb{R}^m$.
- $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ lsc. convex: regularizer.

Work with the metric $\|x\|_M^2 = \langle Mx, x\rangle$, where $M = \lambda^{-1}I - A^*A$.

$$0 < \lambda\|A\|^2 < 1 \Longrightarrow M \text{ is symmetric positive definite.}$$

Apply (IGAHD) to f^M: Moreau envelope of f in the metric M

$$f^M(x) := \min_{\xi \in \mathbb{R}^n} \left\{ f(\xi) + \tfrac{1}{2}\|x - \xi\|_M^2 \right\}.$$

f^M is convex $\mathcal{C}^1$; ∇f^M (in the metric M) is 1-Lipschitz, and

$$\nabla f^M(x) = x - \text{prox}_{\lambda g}\left(x - \lambda A^*(Ax - b)\right).$$

(IGAHD) for (RLS)

Initialize: $x_0 \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^n$

$$\begin{cases} z_k = x_k - \mathrm{prox}_{\lambda g}(x_k - \lambda A^*(Ax_k - b)); \\ y_k = x_k + (1 - \frac{\alpha}{k})(x_k - x_{k-1}) - \beta\sqrt{s}(z_k - z_{k-1}) - \frac{\beta\sqrt{s}}{k} z_k; \\ x_{k+1} = y_k - s\left(y_k - \mathrm{prox}_{\lambda g}(y_k - \lambda A^*(Ay_k - b))\right). \end{cases}$$

Theorem (ACFR, 2019)

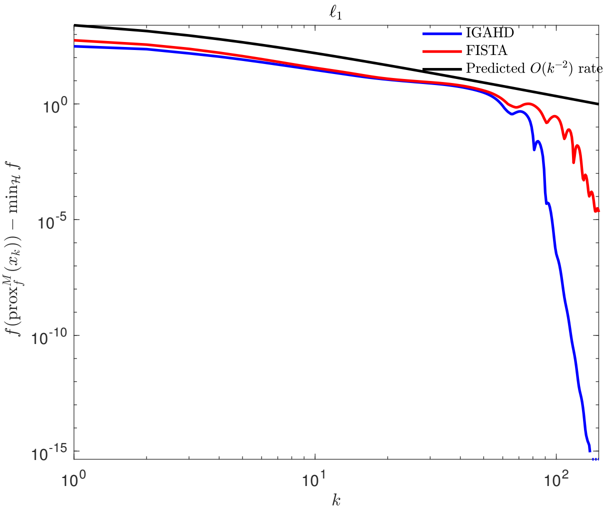
Assumptions: $0 < \lambda \|A\|_2^2 < 1$, $\alpha \geq 3$, $0 \leq \beta < 2\sqrt{s}$, $s \leq 1$.

Let (x_k) *be generated by* (IGAHD) *for* (RLS). *Then,*

$$f(\mathrm{prox}_f^M(x_k)) - \min_{\mathcal{H}} f = \mathcal{O}(k^{-2}), \quad \sum_k k^2 \|\nabla f(x_k)\|^2 < +\infty,$$

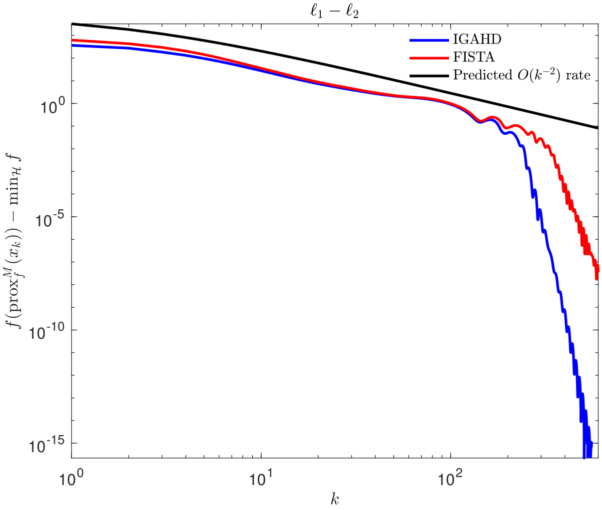
where $\mathrm{prox}_f^M(x_k) := \mathrm{prox}_{\lambda g}(x_k - \lambda A^*(Ax_k - b))$.

Lasso:



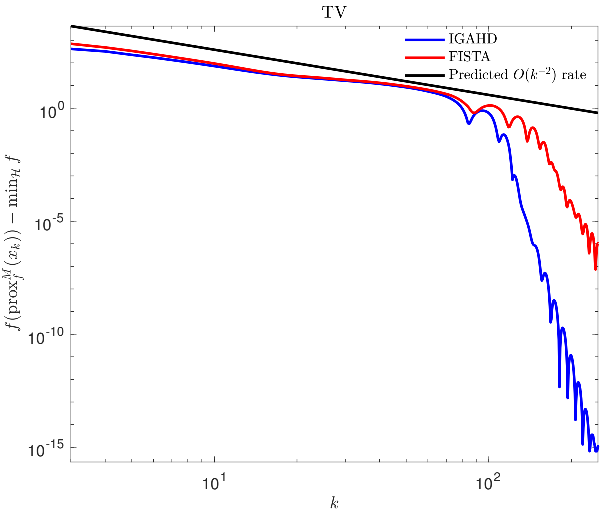
ℓ_1
IGAHD
FISTA
Predicted $O(k^{-2})$ rate
10^0
10^{-5}
10^{-10}
10^{-15}
$f(\mathrm{prox}_f^M(x_k)) - \min_{\mathcal{H}} f$
10^0
10^1
10^2
k

Group Lasso:

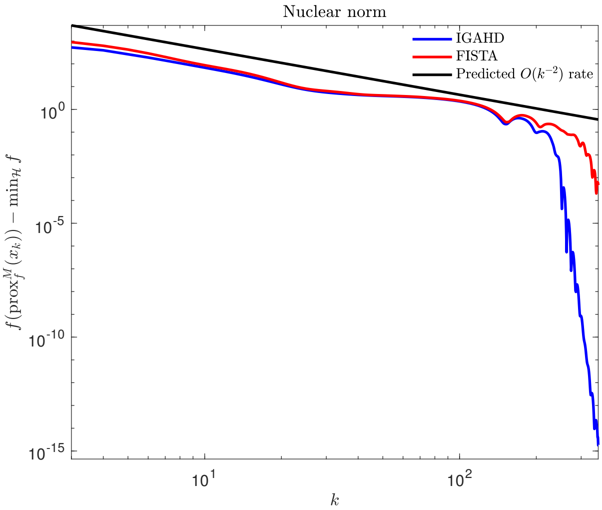


$\ell_1 - \ell_2$
IGAHD
FISTA
Predicted $O(k^{-2})$ rate
10^0
10^{-5}
10^{-10}
10^{-15}
$f(\mathrm{prox}_f^M(x_k)) - \min_{\mathcal{H}} f$
10^0
10^1
10^2
k

TV (total variation):



Nuclear norm:



5. RELATED SYSTEMS

1. First-order in time and space equivalent formulation

Alvarez-A.-Bolte-Redont (JMPA 2002), A.-Peypouquet-Redont (JDE 2016)

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0.$$

$$\Updownarrow$$

$$\begin{cases} \dot{x}(t) + \beta\nabla f(x(t)) - \left(\frac{1}{\beta} - \frac{\alpha}{t}\right)x(t) + \frac{1}{\beta}y(t) = 0; \\ \dot{y}(t) - \left(\frac{1}{\beta} - \frac{\alpha}{t} + \frac{\alpha\beta}{t^2}\right)x(t) + \frac{1}{\beta}y(t) = 0. \end{cases}$$

- Nonsmooth: $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ nonsmooth, (non) convex, proper.
- Damped shocks in mechanics: A.-Maingé-Redont (DEA 2012).
- Numerical applications (temporal discretization):

  Castera-Bolte-Févotte-Pauwels (Deep Learning) (HAL 2019).

  Maingé-Labarre (Fast convergence results, 2020).

2. Alecsa-László-Pinta dynamic model (AMO 2020).

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla f\Big(x(t) + (\gamma + \frac{\beta}{t})\dot{x}(t)\Big) = 0.$$

- Implicit (DIN-AVD): Taylor expansion as $t \to +\infty$, $\dot{x}(t) \to 0$,

$$\nabla f\Big(x(t) + (\gamma + \frac{\beta}{t})\dot{x}(t)\Big) \approx \nabla f(x(t)) + (\gamma + \frac{\beta}{t})\nabla^2 f(x(t))\dot{x}(t).$$

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla f(x(t)) + \beta(t)\nabla^2 f(x(t))\dot{x}(t) = 0.$$

- Explicit temporal discretization $\longrightarrow$ Nesterov accelerated gradient.
- Fast convergence rates:

$$f\Big(x(t) + (\gamma + \frac{\beta}{t})\dot{x}(t)\Big) - \min_{\mathcal{H}} f = \mathcal{O}\left(\frac{1}{t^2}\right).$$

$$\int_{t_0}^{+\infty} t^2 \|\nabla f\Big(x(t) + (\gamma + \frac{\beta}{t})\dot{x}(t)\Big)\|^2 dt < +\infty.$$

- Muehlebach-Jordan (arXiv:1905.07436v1, 2019).

3. Dry friction with Hessian damping, Adly–A. (SIOPT 2020)

$$\ddot{x}(t) + \gamma \dot{x}(t) + \partial\phi(\dot{x}(t)) + \beta \nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) \ni 0.$$

$\phi : \mathcal{H} \to \mathbb{R}_+$ convex, sharp minimum at the origin, $(\phi(x) = r\|x\|)$;

$f : \mathcal{H} \to \mathbb{R}$ differentiable, not necessarily convex, ∇f L-Lipschitz.

Inertial Proximal-Gradient Algorithm

$$y_k = \frac{1}{h(1+h\gamma)}(x_k - x_{k-1}) - \frac{\beta}{1+h\gamma}(\nabla f(x_k) - \nabla f(x_{k-1})) - \frac{h}{1+h\gamma}\nabla f(x_k)$$

$$x_{k+1} = x_k + h\,\mathrm{prox}_{\frac{h}{1+h\gamma}\phi}(y_k).$$

Suppose $\gamma \geq L\left(\frac{h}{2} + \beta\right)$. Then,

- Finite length: $\sum_{k=1}^{+\infty} \|x_{k+1} - x_k\| < +\infty, \quad \lim_{k\to\infty} x_k := x_\infty$.
- Approximate critical point: $-\nabla f(x_\infty) \in \partial\phi(0)$.
- Geometric convergence if $-\nabla f(x_\infty) \in \mathrm{int}(\partial\phi(0))$.
- Tolerates errors not converging to zero: $\|e_k\| \leq r' < r$.

6. MONOTONE INCLUSIONS.

Convergence of inertial dynamics for monotone inclusions

$A : \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone, $J_{\lambda A} = (I + \lambda A)^{-1}$, $A_\lambda = \frac{1}{\lambda}(I - J_{\lambda A})$.

Claim: $x(t) \rightharpoonup x_\infty \in A^{-1}(0)$ as $t \to +\infty$ in the following cases:

$A : \mathcal{H} \to \mathcal{H}$ λ-cocoercive, $\lambda\gamma^2 > 1$, Alvarez-A. (2001), A.-Maingé (2011)

$$\ddot{x}(t) + \gamma \dot{x}(t) + A(x(t)) = 0.$$

$A : \mathcal{H} \to 2^{\mathcal{H}}$ general maximally monotone operator

- A.-Peypouquet (Math. Program. 2019), $\alpha > 2$, $\lambda(t) = (1+\epsilon)\dfrac{t^2}{\alpha^2}$.

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + A_{\lambda(t)}(x(t)) = 0.$$

- A.-László (SIOPT 2020) $\alpha > 1$, $\beta \geq 0$, $\lambda(t) = \lambda t^2$, $\lambda > \frac{1}{(\alpha-1)^2}$.

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta \frac{d}{dt}\left(A_{\lambda(t)}(x(t))\right) + A_{\lambda(t)}(x(t)) = 0.$$

Rate of convergence $\|\dot{x}(t)\| = o\left(1/t\right)$.

Proximal Regularized Inertial Newton Algorithm for Monotone operator

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta \frac{d}{dt}\left(A_{\lambda(t)}(x(t))\right) + A_{\lambda(t)}(x(t)) = 0.$$

Implicit finite-difference scheme: $t_k = kh$, $x_k = x(t_k)$, $\lambda_k = \lambda(t_k)$, $\alpha_k = 1 - \frac{\alpha}{k}$.

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha}{kh^2}(x_k - x_{k-1}) + \frac{\beta}{h}(A_{_k}(x_k) - A_{\lambda_{k-1}}(x_{k-1})) + A_{\lambda_{k+1}}(x_{k+1}) = 0.$$

$$(\text{PRINAM}) \begin{cases} y_k = \left(1 - \beta\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k-1}}\right)\right)x_k + \left(\alpha_k - \frac{\beta}{\lambda_{k-1}}\right)(x_k - x_{k-1}) \\ \qquad\qquad + \beta\left(\frac{1}{\lambda_k}J_{\lambda_k A}(x_k) - \frac{1}{\lambda_{k-1}}J_{\lambda_{k-1}A}(x_{k-1})\right) \\ x_{k+1} = \dfrac{\lambda_{k+1}}{\lambda_{k+1} + s}y_k + \dfrac{s}{\lambda_{k+1} + s}J_{(\lambda_{k+1}+s)A}(y_k). \end{cases}$$

Relaxed proximal inertial algo: A.-Cabot (MP '19), A.-Peypouquet (MP '19).
$A = \partial f$, $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ convex lower semicontinuous proper.

Geometric interpretation of (PRINAM)

$\lim_{\lambda \to +\infty} J_{\lambda A} x = \text{proj}_S(x)$. Hence $J_{(_{k+1}+s)A}(y_k) - y_k \sim \text{proj}_S(y_k) - y_k$.
But only a small step in this direction.

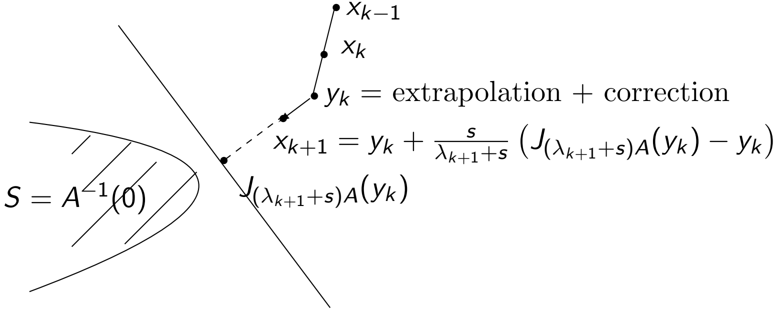


Figure: (PRINAM) algorithm

Theorem (A.-László (SIOPT 2020))

Assumption: $A : \mathcal{H} \to 2^{\mathcal{H}}$ *maximally monotone,* $S = A^{-1}(0) \neq \emptyset$.
$\alpha_k = \frac{t_{k-1}}{t_{k+1}}, t_k = rk + q,\ r > 0, q \in \mathbb{R}$ *and*

$$\lambda_k = \lambda k^2 \ \textit{with} \ \lambda > \frac{(2\beta+s)^2 r^2}{s}.$$

Then, for any sequences (x_k), (y_k) *generated by* (PRINAM)

i) *The speed* $(x_{k+1} - x_k)_{k\geq 1}$ *tends to zero, and*

$$\|x_{k+1} - x_k\| = \mathcal{O}\left(\tfrac{1}{k}\right) \ \textit{as} \ k \to +\infty, \quad \sum_{k\geq 2} k\|x_k - x_{k-1}\|^2 < +\infty$$

$$\|A_{\lambda_k}(x_k)\| = o\left(\frac{1}{k^2}\right) \ \textit{as} \ k \to +\infty, \quad \sum_{k\geq 1} k^3 \|A_{\lambda_k}(x_k)\|^2 < +\infty.$$

ii) *The sequence* (x_k) *converges weakly to some* $\hat{x} \in S$, *as* $k \to +\infty$.

iii) *The sequence* (y_k) *converges weakly to* $\hat{x} \in S$, *as* $k \to +\infty$.
$\|y_k - x_k\| = \mathcal{O}\left(\frac{1}{k}\right)$, *and so* $y_k - x_k$ *converges strongly to zero.*

7. PERSPECTIVE, OPEN QUESTIONS

TAKE AWAY MESSAGE

- A dynamic perspective on accelerated optimization algorithms with viscous and Hessian-driven damping;
- The key is inertia;
- A unified analysis of convergence and integrability;
- New provably accelerated algorithms without explicit Hessian construction.
- Hessian geometric damping neutralizes oscillations; get the best of both world.
- Convergence of trajectories and iterates (Yes);
- Faster asymptotic convergence rates (Yes);
- Inexact/stochastic case (ongoing);
- Operator splitting.

Some open questions concerning Nesterov algorithm

$$\begin{cases} y_k &= x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} &= y_k - s\nabla f(y_k) \end{cases}$$

- Convergence of the iterates in the critical case $\alpha = 3$?
- Optimal tuning of the parameter $\alpha > 3$?
- The sequence (y_k) follows the Ravine method. Is is possible to obtain $1/k^2$ rate of convergence with the Ravine method?
- Is is possible to obtain $1/k^2$ rate of convergence with autonomous dynamic/algorithms?

THANK YOU FOR YOUR ATTENTION

ANY QUESTIONS?

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Annex 1. The Ravine method

In Nesterov accelerated gradient, (y_k) follows the Ravine method.

$$(\mathrm{IG})_\alpha \left\{ \begin{array}{rcl} y_k & = & x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} & = & y_k - s\nabla f(y_k) \end{array} \right.$$

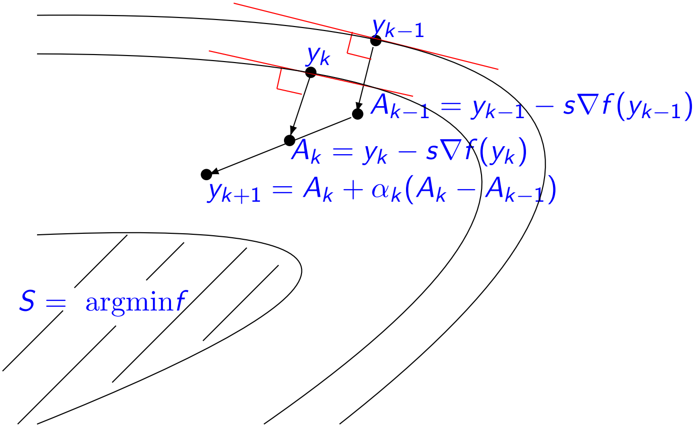
$$y_{k+1} = x_{k+1} + \left(1 - \frac{\alpha}{k+1}\right)(x_{k+1} - x_k)$$

$$= y_k - s\nabla f(y_k) + \left(1 - \frac{\alpha}{k+1}\right)\left(y_k - s\nabla f(y_k) - (y_{k-1} - s\nabla f(y_{k-1}))\right)$$

$$= y_k + \left(1 - \frac{\alpha}{k+1}\right)(y_k - y_{k-1}) - s\nabla f(y_k) - s\left(1 - \frac{\alpha}{k+1}\right)\left(\nabla f(y_k) - \nabla f(y_{k-1})\right)$$

$$(\mathrm{Ravine})_\alpha \left\{ \begin{array}{rcl} A_k & := & y_k - s\nabla f(y_k) \\ y_{k+1} & = & A_k + \left(1 - \frac{\alpha}{k+1}\right)(A_k - A_{k-1}). \end{array} \right.$$

Ravine method: Gelfand, Tsetlin (1961), Nesterov (1983), Polyak ('18).



Shi-Du-Jordan-Su (2018): High-resolution ode, arXiv:1810.08907v3.

Annex 2. General coefficients

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \beta(t)\nabla^2 f(x(t))\dot{x}(t) + b(t)\nabla f(x(t)) = 0.$$

Theorem (A.-Bahlag-Chbani-Riahi,(EECT)), hal-02940534.

Let $x : [t_0, +\infty[\to \mathcal{H}$ *be a solution trajectory of*

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta\nabla^2 f(x(t))\dot{x}(t) + \left(\frac{\beta}{t} + d(t)t^{\alpha-3}\right)\nabla f(x(t)) = 0$$

where $d(\cdot)$ *is a nonincreasing positive function. Then,*

$$a)\ f(x(t)) - \min_{\mathcal{H}} f = \mathcal{O}\left(\frac{1}{t^{\alpha-1}d(t)}\right) \text{ as } t \to +\infty;$$

$$b)\ \int_{t_0}^{+\infty} -\dot{d}(t)t^{\alpha-1}(f(x(t)) - \inf_{\mathcal{H}} f)dt < +\infty;$$

$$c)\ \int_{t_0}^{+\infty} t^{\alpha-1}d(t)\|\nabla f(x(t))\|^2 dt < +\infty.$$