Stochastic optimization with decision-dependent distributions

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One World Optimization Seminar 2020

What this talk is about.

Stochastic optimization with state-dependent distributions

$$\min_{x} \underset{z \sim \mathcal{D}(x)}{\mathbb{E}} [\ell(x, z)] + r(x)$$

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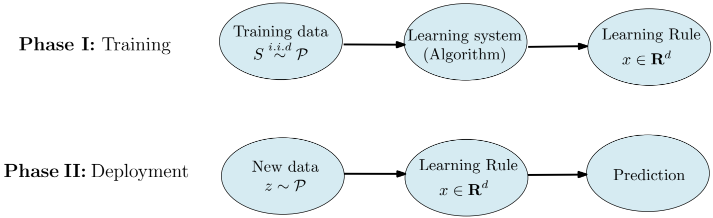
Building on framework Perdomo-Zrnic-Dünner-Hardt:

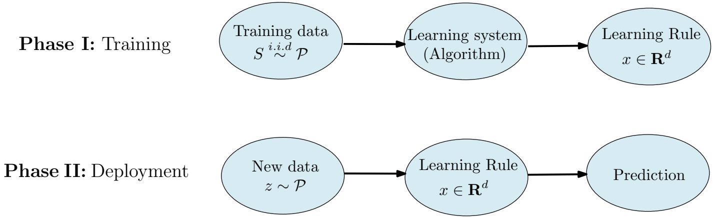
- "Performative prediction" (ICML 2020)
- "Stochastic optimization for performative prediction" (NeurIPS 2020)

Introduction

Pipeline of Supervised Learning

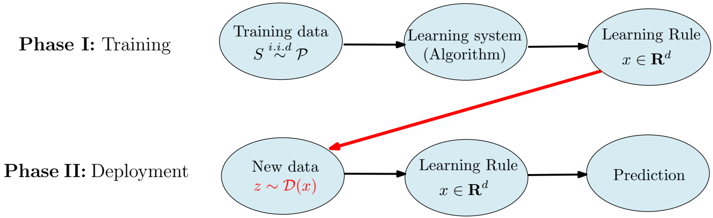


Pipeline of Supervised Learning



Key Assumption: Both test data and training data drawn from $\mathcal{P}$

Learning systems do not exist in isolation...



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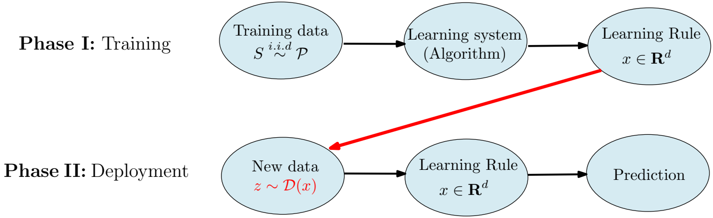

Example (passive interaction):

Bank loan approval influences debt/credit score/#loans.

Example (active interaction): [strategic behavior/gaming]

Individuals alter features to increase likelihood of loan approval.

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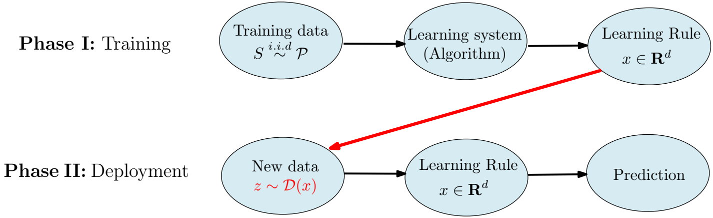

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Perdomo-Zrnic-Dünner-Hardt '20 call this setting **performative prediction**

Optimization model

Stochastic optimization with state-dependent distributions

$$\min_{x} \mathop{\mathbb{E}}_{z \sim \mathcal{D}(x)} [\ell(x, z)] + r(x)$$

where

- $\mathcal{D}(x)$ are state-dependent distributions accessible by sampling
- $\ell(\cdot, z)$ is a convex loss
- $r(\cdot)$ is convex structure-inducing regularizer

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Decision x is judged according to $\mathcal{D}(x)$.

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$$\boxed{\text{Decision } x \text{ is judged according to } \mathcal{D}(x).}$$

Bad news: nonsmooth, nonconvex

Two paths forward:

1. Impose "smoothness" or "structure" on $\mathcal{D}(\cdot)$ and solve.
   e.g. Ahmed '00, Dupačová '06, Goel-Grossman '06, Hassani et al. '20
2. Settle for a related and efficiently computable solution concept.
   Perdomo-Zrnic-Dünner-Hardt '20

Equilibrium

Notation:

$$f_y(x) = \mathop{\mathbb{E}}_{z \sim \mathcal{D}(y)} \ell(x, z) \qquad \text{and} \qquad \nabla f_y(x) = \mathop{\mathbb{E}}_{z \sim \mathcal{D}(y)} \nabla \ell(x, z)$$

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Definition (Perdomo et al '20)

A point $\bar{x}$ is at **equilibrium** for $\mathcal{D}(\cdot)$ if

$$\bar{x} = \operatorname*{argmin}_x \mathop{\mathbb{E}}_{z\sim\mathcal{D}(\bar{x})} \ell(x,z) + r(x)$$

"No incentive to alter $\bar{x}$ based only on response $\mathcal{D}(\bar{x})$."

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Algorithmically: these are fixed points of the map

$$S(y) := \operatorname*{argmin}_x \; f_y(x) + r(x).$$

$\Rightarrow$ suggests a fixed-point algorithm

Performative prediction

Repeated minimization:

$$\boxed{x_{t+1} = \operatorname*{argmin}_{x} \underset{z \sim \mathcal{D}(x_t)}{\mathbb{E}} [\ell(x,z)] + r(x)}$$

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Algorithms for static problems **heuristically** generalize.

Example: Proximal stochastic gradient

$$\begin{Bmatrix} \text{Sample } z_t \sim \mathcal{D}(x_t) \\ \text{Set } x_{t+1} = \operatorname{prox}_{\eta r}(x_t - \eta\nabla\ell(x_t, z_t)) \end{Bmatrix}$$

Similar for dual averaging, prox-point, clipped gradient, fast gradient, ...

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Perdomo et al. '20:

1. Proposed this framework
2. Existence of equilibria
3. Convergence of repeated minimization
4. Convergence of (stochastic) projected gradient method

Our contribution

Meta Thm: *Algorithms that sample according to $\mathcal{D}(x_t)$ can be viewed as the same algorithms applied to the* **static problem**

$$\min_x \underset{z \sim \mathcal{D}(\bar{x})}{\mathbb{E}} [\ell(x, z)] + r(x)$$

where "bias"$\to 0$ linearly as $x_t \to \bar{x}$.

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Recipe:

$$\text{algorithms for static problems} \quad \hookrightarrow \quad \text{“mildly dynamic”}$$

Numerical illustration

Chasing the mean:

$$\min_{x \in \mathbb{R}^2} \mathop{\mathbb{E}}_{z \sim \mathcal{D}(x)} \|x - z\|^2 \qquad \text{where} \qquad \mathcal{D}(x_1, x_2) = N(\rho(x_2, x_1), I)$$

Equilibrium point $\bar{x} = (0, 0)$.

$$\boxed{\nabla f_x(x) \;\; \text{and} \;\; \nabla f_{\bar{x}}(x)}$$

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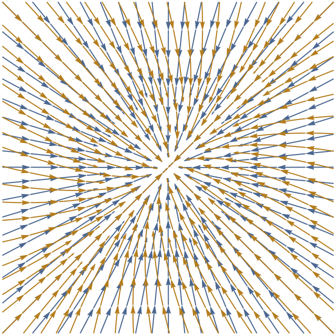

Figure: $\rho = 0.25$

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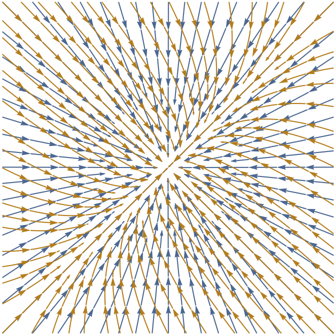

Figure: $\rho = 0.5$

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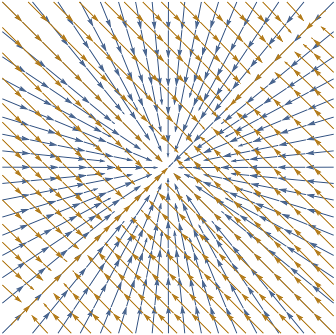

Figure: $\rho = 0.99$

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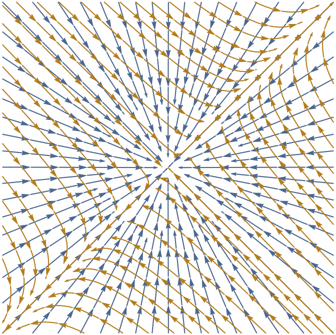

Figure: $\rho = 1.25$

Numerical illustration

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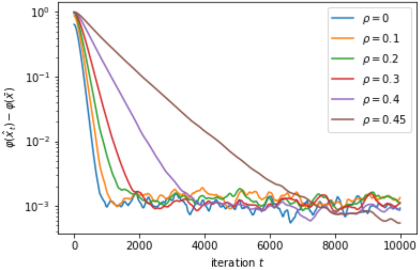

Figure: Stochastic gradient method (fixed $\eta > 0$)

Conclusion: meta-theorem seems valid when $\rho \in (0, 1)$

Outline

- Notation and assumptions
- Two deviation inequalities
- Reduction to online convex optimization
- Stochastic (accelerated) gradient
- Model-based algorithms

Notation:

- Fix $\ell\colon \mathbb{R}^d \times Z \to \mathbb{R}$ where Z is a metric space
- $\mathbb{P} = \{\text{probability measures on } Z\}$ with Wasserstein-1 distance $W_1(\mu, \nu)$

Assumption:

- **(smoothness/convexity)** Loss $\ell(\cdot, z)$ is α-strongly convex and

$$\|\nabla\ell(x,z) - \nabla\ell(x,z')\| \leq \beta \cdot d(z,z')$$
$$\|\nabla\ell(x,z) - \nabla\ell(x',z)\| \leq L \cdot \|x - x'\|$$

- **(sensitivity)** It holds:

$$W_1(\mathcal{D}(x), \mathcal{D}(y)) \leq \gamma \cdot \|x - y\|$$

Conditioning measures:

$$\boxed{\kappa = \frac{L}{\alpha} \qquad \text{and} \qquad \rho = \frac{\gamma\beta}{\alpha}}$$

Interesting regime is $\rho \in (0, 1)$

Recall repeated minimization:

$$x_{t+1} = \underset{x}{\operatorname{argmin}} \ f_{x_t}(x) + r(x)$$

Theorem (Perdomo et al. '20)

If $\rho < 1$, then repeated minimization converges to $\bar{x}$ at linear rate ρ.

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True for wider class of algorithms including proximal point method

$$x_{t+1} = \underset{x}{\operatorname{argmin}} \; f_{x_t}(x) + r(x) + \frac{1}{2\eta}\|x - x_t\|^2$$

Theorem (D-Xiao '20)

If $\rho < 1$, then prox-point method converges to $\bar{x}$ at linear rate $1 - \frac{1-\rho}{1+(\alpha\eta)^{-1}}$.

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Advantage: prox-point is always “distributionally stable”

Regularization experimentally helps!

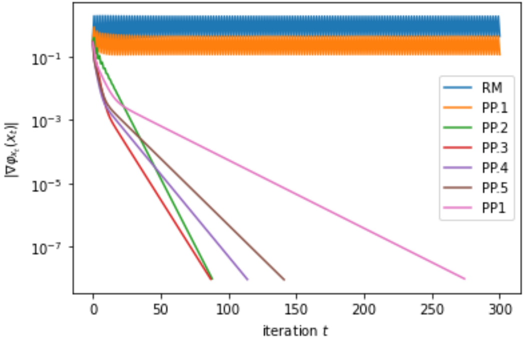

Figure: Strategic classification with $\rho > 1$.

Two deviation inequalities

Define

$$f_y(x) := \mathop{\mathbb{E}}_{z \sim \mathcal{D}(y)} \ell(x, z) \qquad \text{and} \qquad \mathcal{G}_y(x, x') := f_y(x) - f_y(x').$$

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For all $x, y \in \mathbb{R}^d$ it holds:

$$\sup_{x \in \mathbb{R}^d} \|\nabla f_y(x) - \nabla f_{y'}(x)\| \leq \gamma\beta \cdot \|y - y'\|$$

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Implication: $\mathcal{G}_x(x, \bar{x}) - \mathcal{G}_{\bar{x}}(x, \bar{x}) \leq \gamma\beta \cdot \|x - \bar{x}\|^2$ offset by strong convexity

Reduction to online convex optimization

Online convex optimization is a repeated game:

Round $t \geq 1$:

- Player chooses $x_t \in \operatorname{dom} r$
- Nature reveals function ℓ_t and player pays $\ell_t(x_t)$

Player's goal: Minimize the regret

$$R_t := \sum_{i=1}^{t} \big(\ell_i(x_i) + r(x_i)\big) - \min_x \sum_{i=1}^{t} \big(\ell_i(x) + r(x)\big),$$

Algorithms: prox-grad. (Duchi-Singer '09), dual averaging (Xiao '10), Follow-The-Regularized-Leader (FTRL) (McMahan '11)

Guarantees:

$$\left\{ \begin{array}{l} \ell_t \text{ are } \alpha\text{-strongly convex on } \operatorname{dom} r \\ \ell_t \text{ are } G\text{-Lipschitz on } \operatorname{dom} r \end{array} \right\} \quad \Longrightarrow \quad R_t = \mathcal{O}\left(\frac{G^2 \log t}{\alpha}\right)$$

Reduction to online convex optimization

Recall the **equilibrium problem:**

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Theorem (D-Xiao '20)

Suppose $\rho \in (0, \frac{1}{2})$*. Run an online algorithm where in iteration* t*, nature draws* $z_t \sim \mathcal{D}(x_t)$ *and declares* $\ell_t(x_t) = \ell(x_t, z_t)$*. Then*

$$\mathbb{E}\left[\varphi\left(\frac{1}{t}\sum_{i=1}^{t} x_i\right) - \varphi(\bar{x})\right] \ \le \ \frac{\mathbb{E}[R_t]}{(1-2\rho)\,t},$$

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Downside: Requires strong assumptions (bounded domain, Lipschitz loss)

Instead, we analyze algorithms directly.

Assumption: (Finite variance) There is a constant $\sigma > 0$ satisfying

$$\mathop{\mathbb{E}}_{z \sim \mathcal{D}(x)} \left\|\nabla \ell(x, z) - \nabla f_x(x)\right\|^2 \le \sigma^2 \qquad \forall x.$$

Proximal stochastic gradient (SG)

Algorithm:

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Theorem (D-Xiao '20, Dünner '20)

If $\rho < \frac{1}{2}$, proximal SG finds x with $\mathbb{E}[\varphi(x) - \varphi(\bar{x})] \le \varepsilon$ using

$$\mathcal{O}\left(\kappa \cdot \log\left(\frac{\varphi(x_0) - \varphi(\bar{x})}{\varepsilon}\right) + \frac{\sigma^2}{\alpha\varepsilon}\right) \quad \textit{samples.}$$

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If $\rho < 1$, proximal SG finds x with $\|x - \bar{x}\|^2 \leq \varepsilon$ using

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If $\rho < 1$, proximal SG finds x with $\|x - \bar{x}\|^2 \leq \varepsilon$ using

$$\mathcal{O}\left(\kappa \cdot \log\left(\frac{\|x_0 - \bar{x}\|^2}{\varepsilon}\right) + \frac{\sigma^2}{\alpha^2\varepsilon}\right) \quad \textit{samples.}$$

Remark:

1. Reduces to classical rate if $\rho = 0$ (Lan '10)
2. Last iterate convergence if $r = \delta_C$ in (Dünner-Perdomo-Zrnic-Hardt '20)

Proof sketch: grad deviation controls bias

Recall $f_x(x) = \mathbb{E}_{z \sim \mathcal{D}(x)} \ell(x, z) \qquad \Longrightarrow \qquad \texttt{Bias}(x) \;=\; \|\nabla f_x(x) - \nabla f_{\bar{x}}(x)\|$

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$$
\begin{aligned}
\text{Gradient deviation} \quad &\Longrightarrow \quad \texttt{Bias}(x) \leq \beta\gamma \cdot \|x - \bar{x}\| \\
&\Longrightarrow \quad \langle \nabla f_x(x), x - \bar{x} \rangle \geq [f_{\bar{x}}(x) - f_{\bar{x}}(\bar{x})] + \tfrac{\alpha(1-2\rho)}{2} \|x - \bar{x}\|^2
\end{aligned}
$$

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Lemma: (One-step progress) It holds:

$$2\eta \mathbb{E}[\varphi(x_{t+1}) - \varphi(\bar{x})] \le (1 - \hat{\alpha}\eta)\, \mathbb{E}\|x_t - \bar{x}\|^2 - \mathbb{E}\|x_{t+1} - \bar{x}\|^2 + O(\eta^2),$$

where $\hat{\alpha} \approx \alpha(1 - 2\rho)$.

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where $\hat{\alpha} \approx \alpha(1 - 2\rho)$. Combining with strong convexity get

$$\mathbb{E}\|x_{t+1} - \bar{x}\|^2 \leq (1 - \hat{\alpha}\eta)\, \mathbb{E}\|x_t - \bar{x}\|^2 + O(\eta^2),$$

where $\hat{\alpha} \approx \alpha(1 - \rho)$.

... the rest is standard

Proximal accelerated stochastic gradient (ASG)

Algorithm: (Kulunchakov-Mairal '19)

$$\left\{ \begin{array}{l} \text{Sample } z_t \sim \mathcal{D}(y_{t-1}) \text{ and set } g_t = \nabla \ell(y_{t-1}, z_t), \\ \text{Set } x_t = \operatorname{prox}_{\eta_t r}(y_{t-1} - \eta g_t), \\ \text{Set } y_t = x_t + \frac{1-\sqrt{\eta\alpha(1-2\rho)}}{1+\sqrt{\eta\alpha(1-2\rho)}}(x_t - x_{t-1}). \end{array} \right\}$$

Remark: first proximal ASG due to Ghadimi-Lan '13

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Remark: first proximal ASG due to Ghadimi-Lan '13

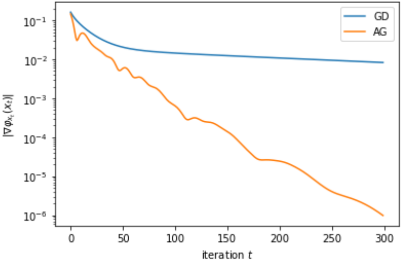
Theorem (D-Xiao '20)

If $\rho \lesssim \kappa^{-1/4}$, proximal ASG finds x satisfying $\mathbb{E}[\varphi(x) - \varphi(\bar{x})] \leq \varepsilon$ using

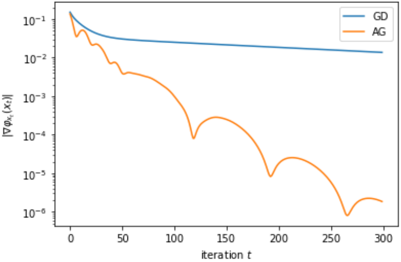
$$\mathcal{O}\left(\sqrt{\kappa} \cdot \log\left(\frac{\varphi(x_0) - \varphi(\bar{x})}{\varepsilon}\right) + \frac{\sigma^2}{\alpha\varepsilon}\right) \qquad \textit{samples.}$$

Proof: technical using stoch. estimate sequences (Kulunchakov-Mairal '19)

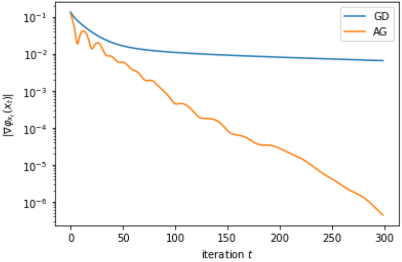
Acceleration works mysteriously well!



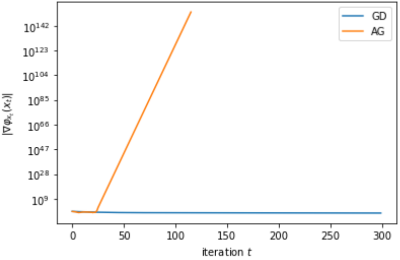
(a) $\gamma = 0$.



(b) $\gamma = 5$.

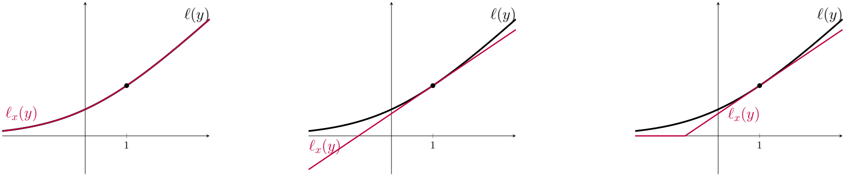


(c) $\gamma = 100$.



(d) $\gamma = 250$.

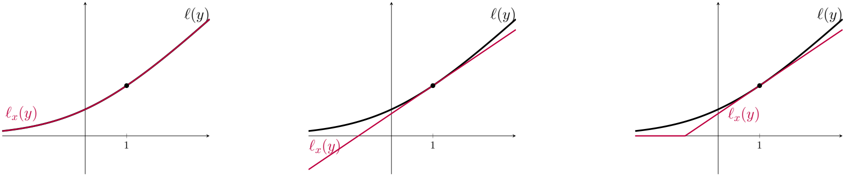
Model-based algorithms



prox−point	gradient	clipped gradient
$\ell_x(y) = \ell(y)$	$\ell_x(y) = \ell(x) + \langle \nabla \ell(x), y - x \rangle$	$\ell_x(y) = (\ell(x) + \langle \nabla \ell(x), y - x \rangle)^+$

- clipped gradient model introduced in Asi-Duchi ’19

Model-based algorithms



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- clipped gradient model introduced in Asi-Duchi '19

Algorithm:

$$\left\{ \begin{array}{l} \text{Sample } z_t \sim \mathcal{D}(x_t) \\ \text{Set } x_{t+1} = \underset{y}{\operatorname{argmin}}\ \ell_{x_t}(y, z_t) + r(y) + \frac{1}{2\eta}\|y - x_t\|^2 \end{array} \right\}$$

Model-based algorithms

Assumption: There exist $\alpha_1, \alpha_2 \geq 0$ such that with $z \sim \mathcal{D}(x)$ have

1. **(Convexity)** $\ell_x(\cdot, z)$ is convex and $\ell_x(\cdot, z) + r$ is α_1-strongly convex.
2. **(Bias/variance)** It holds:

$$\mathop{\mathbb{E}}_{z}[\nabla \ell_x(x,z)] = \nabla f_x(x) \qquad \text{and} \qquad \mathop{\mathbb{E}}_{z} \|\nabla \ell_x(x,z) - \nabla f_x(x)\|^2 \leq \sigma^2.$$

3. **(Accuracy)** The estimates holds:

$$\mathop{\mathbb{E}}_{z}[\ell_x(x,z)] = f_x(x) \qquad \text{and} \qquad \mathop{\mathbb{E}}_{z}[\ell_x(y,z)] + \frac{\alpha_2}{2}\|x-y\|^2 \leq f_x(y).$$

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Remark:

- Similar assumptions in Davis-Drusvyatskiy '19, Asi-Duchi '19
- tighter models yield better algorithms Ryu-Boyd '14, Asi-Duchi '19

Model-based algorithms

Theorem (D-Xiao '20)

If $\frac{\gamma\beta}{\alpha_1+\alpha_2} < \frac{1}{2}$, *algorithm finds* x *with* $\mathbb{E}[\varphi(x) - \varphi(\bar{x})] \leq \varepsilon$ *using*

$$\mathcal{O}\left(\frac{L}{\alpha_1 + \alpha_2} \cdot \log\left(\frac{\varphi(x_0) - \varphi(\bar{x})}{\varepsilon}\right) + \frac{\sigma^2}{(\alpha_1 + \alpha_2)\varepsilon}\right) \qquad \textit{samples.}$$

Model-based algorithms

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If $\frac{\gamma\beta}{\alpha_1+\alpha_2} < 1$*, algorithm finds* x *with* $\|x - \bar{x}\|^2 \leq \varepsilon$ *using*

$$\mathcal{O}\left(\frac{L}{\alpha_1 + \alpha_2} \cdot \log\left(\frac{\|x_0 - \bar{x}\|^2}{\varepsilon}\right) + \frac{\sigma^2}{(\alpha_1 + \alpha_2)^2\varepsilon}\right) \qquad \textit{samples.}$$

Model-based algorithms

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Rates for stochastic proximal point and clipped gradient follow immediately.

Proof sketch: function gap deviation

Lemma: (One-step progress on φ_{x_t}) For every y it holds:

$$2\eta\mathbb{E}[\varphi_{x_t}(x_{t+1})-\varphi_{x_t}(y)] \leq (1-\alpha_2\eta)\mathbb{E}\|x_t-y\|^2-(1+\alpha_1\eta)\mathbb{E}\|x_{t+1}-y\|^2+O(\eta^2),$$

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Gap deviation $\implies$

$$\begin{aligned}\varphi_{x_t}(x_{t+1}) - \varphi_{x_t}(\bar{x}) &\geq \varphi_{\bar{x}}(x_{t+1}) - \varphi_{\bar{x}}(\bar{x}) - \gamma\beta\|x_{t+1} - \bar{x}\| \cdot \|x_t - \bar{x}\| \\ &\geq \varphi_{\bar{x}}(x_{t+1}) - \varphi_{\bar{x}}(\bar{x}) - \frac{\gamma\beta}{2}\|x_{t+1} - \bar{x}\|^2 - \frac{\gamma\beta}{2}\|x_t - \bar{x}\|^2,\end{aligned}$$

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Gap deviation $\implies$

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Combining with Lemma:

$$2\eta\mathbb{E}[\varphi_{\bar{x}}(x_{t+1})-\varphi_{\bar{x}}(y)] \leq (1-\hat{\alpha}_2\eta)\mathbb{E}\|x_t-\bar{x}\|^2-(1+\hat{\alpha}_1\eta)\mathbb{E}\|x_{t+1}-\bar{x}\|^2+O(\eta^2),$$

where $\hat{\alpha}_1 = \alpha_1 - \gamma\beta$ and $\hat{\alpha}_2 = \alpha_2 - \gamma\beta$

... the rest is standard

Inexact repeated minimization (IRM)

In typical applications:

$$\text{“deployment of a learning rule”} \quad \overset{\text{costs}}{\gg} \quad \text{“sampling”}$$

Dünner-Perdomo-Zrnic-Hardt '20:

establish "deployments/samples" trade-off for IRM w/ projected SG method

Theorem (D-Xiao '20)

If $\rho < 1$, can implement IRM with all previous algorithms with same sample efficiency and only $\frac{1}{1-\rho}\log(1/\varepsilon)$ deployments.

Details in the paper:

- "Stochastic optimization with decision-dependent distributions"
D-Xiao (2020), `arxiv.org/abs/2011.11173`

Main references:

- "Performative prediction"
  Perdomo-Zrnic-Dünner-Hardt (ICML 2020)
- "Stochastic optimization for performative prediction"
  Dünner-Perdomo-Zrnic-Hardt (NeurIPS 2020)
- "Strategic classification"
  Hardt-Megiddo-Papadimitriou-Wootters (ACM ITCS '16)

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Thank you!