On the role of interiority notions in convex analysis and optimization

Constantin Zălinescu
Octav Mayer Institute of Mathematics
Iași Branch of Romanian Academy, Iași, Romania
zalinesc@uaic.ro

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Motivation and Aim

The main objects in Convex analysis are the conjugate and the subdifferential of convex functions. Having in view the definition of a convex function, the natural framework is that of real locally convex spaces.

Someone could say that the natural framework is that of linear spaces without topology, and that a topology could be added depending on the problem to be addressed.

This is true, but the framework is not (too) restrictive because always we may endow a linear space with the finest locally convex topology, that is the one induced by all the seminorms defined on it.

So, in the sequel, all the considered linear spaces are real Hausdorff locally convex space if not mentioned explicitly otherwise.

Why consider interiority conditions in Convex analysis and/or convex optimization?

Which are more useful?

Are all necessary?

We try to answer during our talk to these questions, at least partially.

Most problems in convex optimization are of the following type:

(P$_1$) minimize $f(x) + g(Ax)$, s.t. $x \in X$,

(P$_2$) minimize $f(x)$ s.t. $x \in \{u \in X \mid g_1(u) \leq 0, ..., g_m(u) \leq 0\}$,

(P$_3$) minimize $f(x)$ s.t. $x \in \{u \in X \mid G(u) \leq_K 0\}$,

where $f, g_1, ..., g_m : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$ are convex functions, $A \in L(X, Y)$, that is $L : X \to Y$ is a continuous linear operator, $G : X \to Y^\bullet := Y \cup \{\infty_K\}$ is a K-convex function with $K \subset Y$ a convex cone, $y_1 \leq_K y_2$ if $y_1, y_2 \in Y$ are such that $y_2 - y_1 \in K$, and $y \leq_K \infty_K$ for $y \in Y$.

Each of the above problems can be written as

(P) minimize $\Phi(x, 0)$ s.t. $x \in X$

with $\Phi : X \times Y \to \overline{\mathbb{R}}$, where

$\Phi(x, y) := f(x) + g(Ax + y)$ for (P$_1$),

while for (P$_2$) (with $Y := \mathbb{R}^m$) and (P$_3$),

$$\Phi(x, y) := \begin{cases} f(x) & \text{if } g_1(x) \leq y_1, ..., g_m(x) \leq y_m, \\ \infty & \text{otherwise,} \end{cases}$$

$$\Phi(x, y) := \begin{cases} f(x) & \text{if } G(x) \leq_K y, \\ \infty & \text{otherwise,} \end{cases}$$

respectively.

We shall see that using the conjugate and the subdifferential of the function Φ it is possible, under some additional conditions (called, generally, constraint qualification conditions), to have good calculus for these objects for functions obtained by operations which preserve convexity, and to obtain necessary and/or sufficient optimality condition for several types of convex minimization problems.

Notations and some preliminary results on convex functions

- $X, Y, \dots$ – non trivial real Hausdorff locally convex spaces (LCS)
- $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$; f is **convex** if

$$f\left(\lambda x + (1-\lambda)x'\right) \leq \lambda f\left(x\right) + (1-\lambda) f(x') \quad \forall x, x' \in X,\ \lambda \in [0,1]$$

with $(-\infty) + \infty := \infty + (-\infty) := \infty$ and $0 \cdot \infty := \infty$, $0 \cdot (-\infty) := 0$

- $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$ – the **domain** of f
- f is **proper** if $\operatorname{dom} f \neq \emptyset$ and $f(x) \neq -\infty$ for $x \in X$
- $\operatorname{epi} f := \{(x,t) \in X \times \mathbb{R} \mid f(x) \leq t\}$ – the **epigraph** of f
- $\Lambda(X)$ – the class of proper and convex functions defined on X
- $\Gamma(X)$ $(:= \Gamma_\tau(X))$ – the class of (τ-) lower semicontinuous (lsc) convex functions defined on (X, τ)

Clearly,

- f is convex $\Leftrightarrow$ epi f is convex
- If f is convex and $f(x_0) = -\infty$ for some $x_0 \in X$, then $f(x) = -\infty$ for all $x \in \operatorname{icr}(\operatorname{dom} f)$
- f is lsc $\Leftrightarrow$ [epi f is a closed subset of $X \times \mathbb{R}$], $\mathbb{R}^n$ ($n \in \mathbb{N}^*$) being (always) endowed with its usual topology τ_0
- If f is lsc and $f(x_0) = -\infty$ for some $x_0 \in X$, then $f(x) = -\infty$ for all $x \in \operatorname{dom} f$

• $X' := \{\varphi : X \to \mathbb{R} \mid \varphi \text{ linear}\}$ – the **algebraic dual** of X

• $X^* := \{x^* \in X' \mid x^* \text{ continuous}\}$ – the **topological dual** of (X, τ); X^* is endowed with its weakly-star topology w^* if not mentioned explicitly otherwise

• $\langle x, x^* \rangle := x^*(x)$ – when $x \in X$ and $x^* \in X^*$

• $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x) \mid x \in X\}$ – the **conjugate** of $f : X \to \overline{\mathbb{R}}$; f^* is convex and w^*-lsc

• $h^* : X \to \overline{\mathbb{R}}$, $h^*(x) := \sup\{\langle x, x^* \rangle - h(x^*) \mid x^* \in X^*\}$ – the conjugate of $h : X^* \to \overline{\mathbb{R}}$; h^* is convex and lsc

• $\partial_\varepsilon f(x_0) := \{x^* \in X^* \mid \langle x - x_0, x^* \rangle \le f(x) - f(x_0) + \varepsilon \ \forall x \in X\}$ for $f(x_0) \in \mathbb{R}$, $\varepsilon \in \mathbb{R}_+$, and $\partial_\varepsilon f(x_0) := \emptyset$ and $\partial_\varepsilon f(x_0) := \emptyset$ if $f(x_0) \notin \mathbb{R}$ – the ε-**subdifferential** of f at x_0;
$\partial_\varepsilon f(x_0)$ is convex and w^*-closed

• $\partial f(x_0) := \partial_0 f(x_0)$ – the **subdifferential** of f at x_0

Clearly

• $f(x) + f^*(x^*) \geq \langle x, x^* \rangle$ for all $x \in X$, $x^* \in X^*$ – the Fenchel–Young inequality

• $x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \langle x, x^* \rangle + \varepsilon$

• $x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \langle x, x^* \rangle$

For $\emptyset \neq A \subset X$ with X a linear space

• $\operatorname{span} A$, $\operatorname{aff} A$, $\operatorname{conv} A$ – the linear hull, affine hull and convex hull of A

• $\operatorname{lin}_0 A$ – the linear subspace parallel to $\operatorname{aff} A$, that is $\operatorname{lin}_0 A = \operatorname{span}(A - A)$ $(= \operatorname{span}(A - a)\ \forall a \in A)$

Interiority notions

- $\operatorname{cor} A := \{a \in A \mid \forall x \in X,\ \exists \delta > 0,\ \forall \lambda \in [0, \delta] : a + \lambda x \in A\}$ – the **algebraic interior** or **core** of A
- $\operatorname{icr} A := \{a \in A \mid \forall x \in \operatorname{lin}_0 A,\ \exists \delta > 0,\ \forall \lambda \in [0, \delta] : a + \lambda x \in A\}$ – the **relative algebraic interior** or **intrinsic core** of $A \subset X$

For (X, τ) a topological vector space (TVS) and $\emptyset \neq A \subset X$:

- $\operatorname{int} A$ ($:= \operatorname{int}_\tau A$), $\operatorname{cl} A$ ($:= \operatorname{cl}_\tau A$) – the interior and closure of A
- $\operatorname{rint} A$ – the interior of A wrt (the trace topology on) $\operatorname{aff} A$
- $\operatorname{ri} A$ – $\operatorname{rint} A$ if $\operatorname{aff} A$ is closed, and $\emptyset$ otherwise
- $\operatorname{bri} A$ – $\operatorname{icr} A$ if $\operatorname{lin}_0 A$ is barreled wrt its trace topology, and $\emptyset$ otherwise
- $\operatorname{cri} A$ – $\operatorname{icr} A$ if $\operatorname{aff} A$ is closed, and $\emptyset$ otherwise (denoted by ${}^{ic}A$ when A is convex in [Z87][1])

[1] [Z87] C. Z.: Solvability results for sublinear functions and operators, Z. Oper. Res. Ser. A 31 (1987), 79–101.

If, moreover, A is convex:

• $\operatorname{qi} A := \{x \in A \mid \operatorname{cl}(\mathbb{R}_+(A - x)) = X\}$ – the **quasi interior** of A

• $\operatorname{qri} A := \{x \in A \mid \operatorname{cl}(\mathbb{R}_+(A - x))$ is a linear space$\}$ – the **quasi relative interior** of A (see [BL92][2])

• $\operatorname{sqri} A := \operatorname{cri} A$ – the **strong quasi relative interior** of A (see [J90][3])

Some relations among the interiority notions for A convex:

$$\operatorname{aff} A = X \Rightarrow [\operatorname{ri} A = \operatorname{rint} A = \operatorname{int} A \quad \wedge \quad \operatorname{cri} A = \operatorname{icr} A = \operatorname{cor} A],$$
$$\operatorname{cl}(\operatorname{aff} A) = X \Rightarrow \operatorname{qi} A = \operatorname{qri} A,$$
$$\operatorname{aff} A \neq X \Rightarrow [\operatorname{core} A = \operatorname{int} A = \emptyset], \quad \operatorname{cl}(\operatorname{aff} A) \neq X \Rightarrow \operatorname{sqri} A = \emptyset,$$
$$\operatorname{ri} A \subset \operatorname{rint} A \subset \operatorname{icr} A \subset \operatorname{qri} A, \quad [\operatorname{aff} A = \operatorname{cl}(\operatorname{aff} A) \Rightarrow \operatorname{ri} A = \operatorname{rint} A]$$

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[2] [BL92] J.M. Borwein, A. Lewis: Partially finite convex programming, Part I: Quasi relative interiors and duality theory, Math. Program. 57 (1992), 15–48] (preprint version in 1987 or 1988).

[3] [J90] V. Jeyakumar, Duality and infinite dimensional optimization, Nonlinear Analysis, TMA, 15 (1990), 1111–1122.

The properties of convex sets with nonempty interior are well known, and those for int replaced by cor, icr, rint, ri, bri and cri are similar (being themselves interiors of convex sets wrt to certain linear topologies).

We also introduce the conditions (Hx) and (Hwx) below, where Y is another TVS and $A \subset X \times Y$, "x" in (Hx), (Hwx) referring to the component $x \in X$:

(Hx) If the sequences $\left((x_n, y_n)\right)_{n\geq 1} \subset A$ and $(\lambda_n)_{n\geq 1} \subset \mathbb{R}_+$ are such that $\sum_{n\geq 1} \lambda_n = 1$, $\sum_{n\geq 1} \lambda_n y_n$ has sum y and $\sum_{n\geq 1} \lambda_n x_n$ is Cauchy, then the series $\sum_{n\geq 1} \lambda_n x_n$ is convergent and its sum $x \in X$ verifies $(x, y) \in A$.

(Hwx) If the sequences $\left((x_n, y_n)\right)_{n\geq 1} \subset A$ and $(\lambda_n)_{n\geq 1} \subset \mathbb{R}_+$ are such that $\left((x_n, y_n)\right)$ is bounded, $\sum_{n\geq 1} \lambda_n = 1$, $\sum_{n\geq 1} \lambda_n y_n$ has sum y, then the series $\sum_{n\geq 1} \lambda_n x_n$ is convergent and its sum $x \in X$ verifies $(x, y) \in A$.

The interest for these notions (less sqri and qri) and conditions is given by the following "open mapping theorems", in which X is a LCS and Y is a TVS:
The next result is Simons theorem established in [S90][4]

Theorem 1 (Simons)

Let X and Y be first countable and $\Gamma : X \rightrightarrows Y$. Assume that X is a locally convex space, $\operatorname{gph}\Gamma$ satisfies condition (Hwx), $y_0 \in \operatorname{bri}(\operatorname{Im}\Gamma)$ and $x_0 \in \Gamma^{-1}(y_0)$. Then $y_0 \in \operatorname{rint}\Gamma(U)$ for every $U \in \mathcal{N}_X(x_0)$. In particular $\operatorname{bri}(\operatorname{Im}\Gamma) = \operatorname{rint}(\operatorname{Im}\Gamma)$ if $\operatorname{bri}(\operatorname{Im}\Gamma) \neq \emptyset$.

[4] [S90] S. Simons: The occasional distributivity of $\circ$ over $\overset{e}{+}$ and the change of variable formula for conjugate functions, Nonlinear Anal. 14 (1990), 1111–1120.

The next result is Ursescu theorem established in [U75][5]; when X and Y are Banach spaces and $\operatorname{aff}(\operatorname{Im}\Gamma) = Y$, this result is known as Robinson–Ursescu theorem.

Theorem 2 (Ursescu)

Let X be a complete semi-metrizable locally convex space and $\Gamma : X \rightrightarrows Y$ be a closed convex multifunction. Assume that $y_0 \in \operatorname{bri}(\operatorname{Im}\Gamma)$ and $x_0 \in \Gamma^{-1}(y_0)$. Then $y_0 \in \operatorname{rint}\Gamma(U)$ for every $U \in \mathcal{N}_X(x_0)$. In particular $\operatorname{bri}(\operatorname{Im}\Gamma) = \operatorname{rint}(\operatorname{Im}\Gamma)$ if $\operatorname{bri}(\operatorname{Im}\Gamma) \neq \emptyset$.

[5] [U75] C. Ursescu: Multifunctions with convex closed graph, Czechoslovak Math. J. 25(100) (1975), 438–441.

Perturbed minimization problems

Let X, Y be TVSs, $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a function and

$$h : Y \to \overline{\mathbb{R}}, \qquad h(y) := \inf\{\Phi(x,y) \mid x \in X\}$$

be the **marginal function** associated to Φ; h is also called the *value* or *performance* function associated to Φ. It is easy to verify that $\operatorname{dom} h = \Pr_Y(\operatorname{dom}\Phi)$ and h is convex when Φ is so.

It is useful to consider the following (minimization) problem

$(P) \qquad \min \ \Phi(x,0), \ x \in X,$

called the **primal problem,** and the following one, called the **dual problem** of (P):

$(D) \qquad \max \ (-\Phi^*(0,y^*)), \ y^* \in Y^*.$

It is obvious that (D) is equivalent to the convex programming problem

$$(D') \qquad \min \Phi^*(0, y^*), \ y^* \in Y^*.$$

The equivalence has to be understood in the sense that the problems (D) and (D') have the same (ε-)solutions; moreover $v(D') = -v(D)$ (of course, for a maximization problem the notions of (ε-)solution, local solution and value are defined dually to those for minimization problems).

It is nice to observe that (D') and (P) are of the same type.

In the following results we mention some properties which connect the problems (P), (D) and the function h. Most of these assertions can be found in I. Ekeland and R. Temam's book [ET74][6]; its statemnt is quoted from [Z02].[7]

[6] [ET74] I. Ekeland, R. Temam: *Analyse convexe et problèmes variationnels,* Dunod, Paris (1974).

[7] [Z02] C. Z.: Convex Analysis in General Vector Spaces, World Scientific, River Edge, NJ (2002).

Theorem 3 (to be continued)

Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ and $h : Y \to \overline{\mathbb{R}}$ be the marginal function associated to Φ. Then:

(i) $h^*(y^*) = \Phi^*(0, y^*)$ for every $y^* \in Y^*$.

(ii) Let $(\overline{x}, \overline{y}) \in X \times Y$ be such that $\Phi(\overline{x}, \overline{y}) \in \mathbb{R}$. Then

$$(0, y^*) \in \partial\Phi(\overline{x}, \overline{y}) \ \Leftrightarrow \ h(\overline{y}) = \Phi(\overline{x}, \overline{y}) \text{ and } y^* \in \partial h(\overline{y}).$$

(iii) $v(P) = h(0)$ and $v(D) = h^{**}(0)$. Therefore $v(P) \geq v(D)$; hence **weak duality** holds.

(iv) Suppose that Φ is proper, $\overline{x} \in X$ and $\overline{y}^* \in Y^*$. Then $(0, \overline{y}^*) \in \partial\Phi(\overline{x}, 0) \Leftrightarrow [\overline{x}$ is a solution of problem (P), $\overline{y}^*$ is a solution of (D) and $v(P) = v(D) \in \mathbb{R}]$.

Theorem 3 (continued)

Assume, moreover, that Φ is convex. Then

(v) $[h(0) \in \mathbb{R}$ and h is lsc at $0] \Leftrightarrow v(P) = v(D) \in \mathbb{R}$; hence, in this case, **strong duality** holds;

(vi) $[h(0) \in \mathbb{R}$ and $\partial h(0) \neq \emptyset] \Leftrightarrow [v(P) = v(D) \in \mathbb{R}$ and (D) has optimal solutions]. In this situation $S(D) = \partial h(0)$; hence (P) is **stable** in this case.

(vii) Furthermore, assume that Φ is proper. TFAE: (a) $\overline{h}$ is proper; (b) h^* is proper]; (c) h is minorized by an affine continuous functional; (c)

$$\exists\, \overline{y}^* \in Y^*,\ \exists\, \alpha \in \mathbb{R},\ \forall (x,y) \in X \times Y \ :\ \Phi(x,y) \geq \langle y, \overline{y}^* \rangle + \alpha,$$

where $\overline{h} : Y \to \overline{\mathbb{R}}$ is such that $\operatorname{epi} \overline{h} := \operatorname{cl}(\operatorname{epi} h)$.

In the sequel we are interested only by the case in which Φ is convex.

So, let $\Phi \in \Lambda(X \times Y)$ be such that $0 \in \operatorname{dom} h = \Pr_Y(\operatorname{dom}\Phi)$; set

$$Y_0 := \operatorname{aff}\left(\Pr_Y(\operatorname{dom}\Phi)\right) = \operatorname{lin}_0\left(\Pr_Y(\operatorname{dom}\Phi)\right) = \operatorname{lin}_0(\operatorname{dom} h).$$

It is known that $h^{**} = -\infty$ if $\overline{h}$ is not proper, and $\overline{h} = h^{**}$ if $\overline{h}$ is proper (see, e.g., [Z02, Th. 2.3.4]).

The most important case is when (P) is stable, that is, when $h(0) \in \mathbb{R}$ and $\partial h(0) \neq \emptyset$, then the case in which strong duality holds, that is, $\overline{h}(0) = h(0) \in \mathbb{R}$.

Because h is a convex function, the following results can be applied for the subdifferentiability of h at 0:

Proposition 4

Let (X, τ) be a TVS and $f : X \to \overline{\mathbb{R}}$ a convex function. TFAE:
(a) there exists $x_0 \in \operatorname{dom} f$ such that f is continuous at x_0; (b) f is bounded above on some nonempty open subset of $\operatorname{dom} f$;
(c) $\operatorname{int}(\operatorname{epi} f) \neq \emptyset$.

Proposition 5

Let (X, τ) be a LCS and let the convex function $f : X \to \overline{\mathbb{R}}$ and $x_0 \in X$ be such that $f|_{X_0}$ is continuous and finite at x_0, where $X_0 := \operatorname{aff}(\operatorname{dom} f)$ is endowed with the trace topology. Then $\partial f(x_0) \neq \emptyset$.

In the case in which $f|_{X_0}$ is continuous and finite at x_0, automatically $\operatorname{icr}(\operatorname{dom} f) \neq \emptyset$, but there are many situations (e.g., entropy optimization problems, when X is often $L_p(I)$ or ℓ_p with $p \in [1, \infty[$) when such a condition is not satisfied.

Having in view that $A := \Pr_Y(\operatorname{epi}\Phi)$ $(\subset Y \times \mathbb{R})$ is a set of *epigraph type*, that is, $A = A + \{0\} \times \mathbb{R}_+$, and $h(y) = \varphi_A(y) := \inf\{t \in \mathbb{R} \mid (y, t) \in A\}$ for $y \in Y$, in such a situation, the following result (see [Z15, Prop. 8][8]) could be applied:

Proposition 6

Let $A \subset Y \times \mathbb{R}$ be a nonempty convex set of epigraph type and let $\overline{y} \in \Pr_Y(A) = \operatorname{dom}\varphi_A$ be such that $\overline{\alpha} := \varphi_A(\overline{y}) \in \mathbb{R}$.

(i) If $\overline{y} \in \operatorname{qri}(\operatorname{cl}(\Pr_Y(A)))$ and $(\overline{y}, \overline{\alpha}) \notin \operatorname{qri}(\operatorname{cl} A)$, then $\partial\varphi_A(\overline{y}) \neq \emptyset$.

(ii) If $\partial\varphi_A(\overline{y}) \neq \emptyset$ then $(\overline{y}, \overline{\alpha}) \notin \operatorname{qri}(\overline{\operatorname{cl}} A)$.

[8][Z15] C. Z.: *On the use of the quasi-relative interior in optimization*, Optimization 64 (2015), 1795–1823.

Of course, from the above result one deduces rapidly Proposition 5. Observe that assertions (i) and (ii) are very close to each other, and both are close to the subdifferentiability of φ_A at $\overline{y}$, in order to be very effective.

This is also seen from the following properties of the quasi relative interior, where $\emptyset \neq C \subset X$ is convex:
$\operatorname{qri} C = C \cap \operatorname{qri}(\operatorname{cl} C)$ and

$$x_0 \in C \setminus \operatorname{qri} C \Leftrightarrow [\exists x^* \in X^* : \sup x^*(C) > \inf x^*(C) = \langle x_0, x^* \rangle]$$

The relation (FDF) in the conclusion of the following theorem is very useful for obtaining important results in Convex analysis and convex programming.

Theorem 7 (to be continued)

Let $\Phi \in \Lambda(X \times Y)$ be such that $0 \in C := \Pr_Y(\operatorname{dom} \Phi)$ $(= \operatorname{dom} h)$ and $h(0) \in \mathbb{R}$. Consider $Y_0 := \operatorname{span} C$. Suppose that one of the following conditions is satisfied:

(i) there exists $\lambda_0 \in \mathbb{R}$ such that $V_0 := \{y \in Y \mid \exists x \in X, \ \Phi(x, y) \leq \lambda_0\} \in \mathcal{N}_{Y_0}(0)$;

(ii) there exist $\lambda_0 \in \mathbb{R}$ and $x_0 \in X$ such that

$$\forall \, U \in \mathcal{N}_X \ : \ \{y \in Y \mid \exists x \in x_0 + U, \ \Phi(x, y) \leq \lambda_0\} \in \mathcal{N}_{Y_0}(0);$$

(iii) there exists $x_0 \in X$ such that $(x_0, 0) \in \operatorname{dom} \Phi$ and $\Phi(x_0, \cdot)$ is continuous at 0;

(iv) X and Y are metrizable, $\operatorname{epi} \Phi$ satisfies condition *(Hwx)* and $0 \in \operatorname{bri} C$;

(v) X is a Fréchet space, Y is metrizable, Φ is a li-convex function and $0 \in \operatorname{bri} C$;

Theorem 7 (continued)

(vi) X is a Fréchet space, Φ is lsc and $0 \in \operatorname{bri} C$;

(vii) X, Y are Fréchet spaces, Φ is lsc and $0 \in \operatorname{sqri} C$;

(viii) $\dim Y_0 < \infty$ and $0 \in \operatorname{icr} C$;

(ix) there exists $x_0 \in X$ such that $\Phi(x_0, \cdot)$ is quasi-continuous and the sets$\{0\}$, C are united,

(x) $0 \in \operatorname{qri} C$ and $(0, h(0)) \notin \operatorname{qri}(\operatorname{cl}(\operatorname{Pr}_{Y\times\mathbb{R}}(\operatorname{epi}\Phi)))$.

Then (P) is stable, that is,

$$\inf_{x\in X} \Phi(x,0) = \max_{y^*\in Y^*} \big(-\Phi^*(0,y^*)\big); \tag{FDF}$$

moreover, if one of the conditions (i)–(ix) holds, then $h|_{Y_0}$ is continuous at 0. Furthermore, $\overline{x} \in X$ is a minimum point for $\Phi(\cdot,0)$ if and only if there exists $\overline{y}^* \in Y^*$ such that $(0,\overline{y}^*) \in \partial\Phi(\overline{x},0)$.

The function $g \in \Lambda(Y)$ is *quasi-continuous* when $Y_1 := \operatorname{aff}(\operatorname{dom} g)$ is closed, has finite codimension and $g|_{Y_1}$ is continuous on $\operatorname{rint}(\operatorname{dom} g)$ (assumed to be nonempty).

The convex sets $\emptyset \neq A, B \subset X$ are *united* if A and B cannot be properly separated by a closed hyperplane.

Observe that (FDF) is true when $h(0) = -\infty$, without any condition on the involved functions, because $h^*(y^*) = \infty$ for all $y^* \in Y^*$.

Sketch of the proof:

By Proposition 6 one has $\partial h(0) \neq \emptyset$ when (x) holds, and so (FDF) holds by Theorem 24.

If (i) holds, then $h|_{Y_0}$ is continuous at 0, being bounded above by λ_0 on V_0, and so $\partial h(0) \neq \emptyset$.

(ii) $\Rightarrow$ (i) is obvious;

(iii) $\Rightarrow$ (ii) One takes $\lambda_0 := \Phi(x_0, 0) + 1$; then $V_0 := [\Phi(x_0, \cdot) \leq \lambda_0] \in \mathcal{N}_Y(0)$ (hence $Y_0 = Y$). Then $V_0 \subset \{y \mid \exists x \in x_0 + U : \Phi(x_0, y) \leq \lambda_0\}$ $\forall U \in \mathcal{N}_X$.

(iv) $\Rightarrow$ (ii) One takes $\Gamma : X \times \mathbb{R} \rightrightarrows Y$ with $\operatorname{gph} \Gamma := \{(x, t, y) \mid (x, y, t) \in \operatorname{epi} \Phi\}$; then Γ satisfies (Hw(x, t)), then one applies Simons' theorem.

(v) $\Rightarrow$ (ii), (vi) $\Rightarrow$ (ii) The proofs are similar the preceding one, using Simons theorem for the first implication (for another multifunction) and Ursescu theorem for the other one.

(vii) implies (iv), (v) and (iv).

(viii) implies the continuity of $h|_{Y_0}$ (hence its differentiability) on icr(dom h) because dim $Y_0 < \infty$.

(ix) implies the continuity of $h|_{Y_0}$ on rint(dom h) ($=$ icr(dom h)) by [Z02, Props. 2.2.15, 2.1.8].

Condition (iii) is classical for having (FDF), and can be found in almost all books and articles dealing with perturbation functions; condition (vii) is considered in this form in [Z87, Th. 6] and in [J90, Prop. 3.1]; condition (x) is considered in [Z15, Prop. 19]; for detailed historical notes on the other conditions see [Z99, Rem. 2][9].

Taking $\Phi(x, y) := \begin{cases} f(x) & \text{if } g_1(x) \leq y_1, ..., g_m(x) \leq y_m, \\ \infty & \text{otherwise,} \end{cases}$
where $f, g_1, ..., g_m \in \Lambda(X)$ are as in problem (P_2), one get $\Phi^*(0, -\lambda) = \sup_{x \in X} [-L(x, \lambda)]$ for $\lambda \in \mathbb{R}^m_+$, and $\Phi^*(0, -\lambda) = \infty$ otherwise. So, (FDF) becomes $v(P_2) = \max_{\lambda \geq 0} \inf_{x \in X} L(x, \lambda)$, and so Theorem 24 provides sufficient conditions for the existence of Lagrange multipliers for (P_2), the simplest one being the Slater condition (there exist $x_0 \in \operatorname{dom} f$ such that $g_k(x_0) < 0$ for $k \in \overline{1, m}$), provided by condition (iii).

[9] [Z99] C. Zălinescu: *A comparison of constraint qualifications in infinite dimensional convex programming revisited*, J. Austral. Math. Soc. B 40 (1999), 353–378.

In (other) applications, it is important to have conditions on Φ which also ensure that the functions $\widetilde{\Phi}$, $\widetilde{\Phi}(x, y) := \Phi(x, y) - \langle x, x^* \rangle$ with $x^* \in X^*$, satisfy some of the conditions mentioned in Theorem 7. Such conditions are conditions (ii)–(ix) of Theorem 7, but this is not always true for (i); we do not know if it is true or not for (x).

The next result is an instance of application of Theorem 7.

Theorem 8 (to be continued)

Let $f \in \Lambda(X)$, $g \in \Lambda(Y)$ and $A \in \mathcal{L}(X, Y)$ be such that $0 \in C := A(\mathrm{dom}\, f) - \mathrm{dom}\, g$; set $Y_0 :=$ span C. Consider $\varphi \in \Lambda(X)$ with $\varphi(x) := f(x) + g(Ax)$ for $x \in X$. Assume that one of the following conditions holds:

(i) there exist $\lambda_0 \in \mathbb{R}$, $B \in \mathcal{B}_X$ and $V_0 \in \mathcal{N}_{Y_0}$ such that

$$V_0 \subset A\left([f \leq \lambda_0] \cap B\right) - [g \leq \lambda_0];$$

(ii) for every $U \in \mathcal{N}_X$ there exist $\lambda > 0$ and $V \in \mathcal{N}_{Y_0}$ such that

$$V \subset A\left([f \leq \lambda] \cap \lambda U\right) - [g \leq \lambda];$$

(iii) there exists $x_0 \in \mathrm{dom}\, f \cap A^{-1}(\mathrm{dom}\, g)$ such that g is continuous at Ax_0;

(iv) X, Y are metrizable, $0 \in$ bri C, f and g have proper conjugates, either f, g are cs-closed and gph A is cs-complete, or f, g are cs-complete;

Theorem 8 (continued)

(v) X is a Fréchet space, Y is metrizable, f, g are li-convex functions and $0 \in \operatorname{bri} C$;

(vi) X is a Fréchet space, f, g are lsc and $0 \in \operatorname{bri} C$;

(vii) X, Y are Fréchet spaces, f, g are lsc and $0 \in \operatorname{sqri} C$;

(viii) $\dim Y_0 < \infty$ and $0 \in \operatorname{icr} C$;

(ix) g is quasi-continuous and $A(\operatorname{dom} f)$ and $\operatorname{dom} g$ are united;

(x) $Y = \mathbb{R}^n$, $\mathrm{qri}(\mathrm{dom}\, f) \neq \emptyset$ and $A\big(\mathrm{qri}(\mathrm{dom}\, f)\big) \cap \operatorname{icr}(\mathrm{dom}\, g) \neq \emptyset$.

Then for every $x^* \in X^*$, $x \in \operatorname{dom} \varphi$ and $\varepsilon \geq 0$ we have:

$$\varphi^*(x^*) = \min\{f^*(x^* - A^* y^*) + g^*(y^*) \mid y^* \in Y^*\}, \qquad (*)$$

$$\partial_\varepsilon \varphi(x) = \bigcup \{\partial_{\varepsilon_1} f(x) + A^*(\partial_{\varepsilon_2} g(Ax)) \mid \varepsilon_1, \varepsilon_2 \geq 0,\ \varepsilon_1 + \varepsilon_2 = \varepsilon\},$$

$$\partial \varphi(x) = \partial f(x) + A^*(\partial g(Ax)).$$

In what concerns the proof, it is sufficient to prove the (good) formula (*) for φ^*; having (*), one gets easily the formula for $\partial_\varepsilon \varphi(x)$ using the fact that $x^* \in \partial_\varepsilon \varphi(x)$ iff $(\langle x, x^* \rangle \leq)$ $\varphi(x) + \varphi^*(x^*) \leq \langle x, x^* \rangle + \varepsilon$.

In fact J.-B. Hiriart-Urruty (see [H82][10]) used such good formulae for the conjugates of $f_1 + f_2$, $g \circ A$, $\max\{f_1, ..., f_m\}$, etc. for getting their ε-subdifferentials.

Because the inequality
$\varphi^*(x^*) \leq \inf\{f^*(x^* - A^* y^*) + g^*(y^*) \mid y^* \in Y^*\}$ is true without any supplementary condition on f, g, A, it is sufficient to take $x^* \in \operatorname{dom} \varphi^*$ and to prove the existence of some $y^* \in Y^*$ such that $\varphi^*(x^*) = f^*(x^* - A^* y^*) + g^*(y^*)$. For this, one considers $\Phi : X \times Y \to \overline{\mathbb{R}}$ defined by $\Phi(x, y) := f(x) + g(Ax - y) - \langle x, x^* \rangle$.

[10] [H82] J.-B. Hiriart-Urruty: *ε-subdifferential calculus*, In: Convex analysis and optimization (London, 1980), Pitman, Boston, pp. 43–92 (1982).

One verifies easily that the corresponding conditions from the preceding theorem are verified by this Φ, and so the existence of the desired y^* is provided by (FDF).

For $x^* := 0$, the relation (*) becomes the well known Fenchel–Rockafellar duality formula:

$$\inf\{f(x)+g(Ax) \mid x \in X\} = \max\{-f^*(A^*y^*)-g^*(-y^*) \mid y^* \in Y^*\}. \tag{**}$$

For $Y := \mathbb{R}^n$, (**) is proved in [BL, Th. 4.2(i), Cor. 4.3] under conditions (viii) and (x), respectively. Applying Theorem 8 under condition (x) for $\Phi(x,y) := f(x) + g(Ax - y)$ (hence $x^* = 0$), one obtains [BCW08, Th. 3.14][11].

For detailed historical notes on the conditions (i)–(x) from Theorem 8 see [Z99, Rem. 9].

[11] [BCW08] R.I. Boţ, E.R. Csetnek, G. Wanka: *Regularity conditions via quasi-relative interior in convex programming*, SIAM J. Optim. 19 (2008), 217–233.

In [GT90][12], M.S. Gowda and M. Teboulle discussed several sufficient conditions for the validness of (**) when f, g are lsc and X, Y are Banach spaces. More precisely, setting $C := A(\operatorname{dom} f) - \operatorname{dom} g$, one discusses (and compares) the conditions (S) $0 \in \operatorname{int} C$, (R) $0 \in \operatorname{cor} C$, (RR) $0 \in \operatorname{ri} C$ when $Y := \mathbb{R}^n$, (GCQ) $0 \in \operatorname{sqri} C$, the last one being the most general; however, for the proof of (**) under (GCQ), one reduces the problem to one in which (R) is satisfied using a similar argument to the standard proof of Proposition 4.

When X $(= Y)$ is a Banach space and $A := \operatorname{Id}_X$, that is the case of the sum of two functions, condition (vii) is nothing else than the Attouch–Brezis condition for (**) from [AB86, Th. (1.1)][13].

[12][GT90] M.S. Gowda, M. Teboulle: *A comparison of constraint qualifications in infinite-dimensional convex programming*, SIAM J. Control Optim. 28 (1990), 925–935.

[13][AB86] H. Attouch, H. Brezis, *Duality for the sum of convex functions in general Banach spaces*, in Aspects of Mathematics and its Applications, J. A. Barroso, ed., pp. 125–133, Elsevier Science, Amsterdam (1986).

Thank you for your attention!