Revisiting Augmented Lagrangian Duals

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eprints for the optimization community

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CEPID - Center for Mathematical Sciences Applied to Industry

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IMECC-UNICAMP Brazil

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Context: when to use decomposition methods?

- problems too difficult to solve directly
- problems with partial separable structure
- problems of problems
- information not accesible

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- information not accesible **in ML, commercial oracles**

Which decomposition method?

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Decomposition: what and how?

Primal

Decomposition: what and how?

Primal

Dual

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Primal-Dual

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It all depends on the output of interest

Illustration with a simple example

$$\begin{cases} \min & f_T(y_T) + f_H(y_H) \\ \text{s.t.} & y_T \in \mathcal{S}_T \,,\, y_H \in \mathcal{S}_H \\ & \\ & \end{cases}$$

Two power plants

$y_T \in \mathcal{S}_T$

$f_T(y_T)$

$y_H \in \mathcal{S}_H$

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$\boxed{y_T + y_H = d}$ (demand)

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$\langle \mathcal{F}, x \rangle + f_T(y_T)$

$x \in \{0, 1\}$ and $y_T \leq x\, y^{up}$

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$$\left\{\begin{array}{ll} \min & \langle \mathcal{F}, x\rangle + f_T(y_T) + f_H(y_H) \\ \text{s.t.} & y_T \in \mathcal{S}_T\,, y_H \in \mathcal{S}_H \\ & y_T + y_H = d \\ & x \in \{0,1\} \text{ and } y_T \le x\, y^{up} \iff (x, y_T) \in \mathcal{S}_T \end{array}\right.$$

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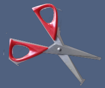
Primal scissors (Benders)

- **Good**, to split 0-1 variables from possible NLP relations



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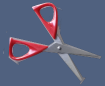

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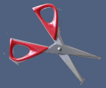

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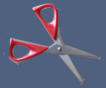
$\Longrightarrow$ operational subproblem in both y_T, y_H

for each given x_k (a master program defines x_{k+1})



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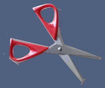
To separate technologies

we need **dual** scissors



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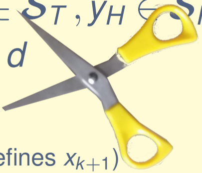

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acting on the demand constraint

Dual scissors: Lagrangian relaxation

$$\left\{\begin{array}{rl} \min & f_T(x, y_T) + f_H(y_H) \\ \text{s.t.} & (x\,, y_T) \in \mathcal{S}_T \\ & y_H \in \mathcal{S}_H \\ & y_T + y_H = d \qquad \leftrightarrow \mathbf{u} \end{array}\right. \Longrightarrow$$

$$\begin{aligned} L(x, y, u) \quad = \quad & f_T(x, y_T) + f_H(y_H) \\ & + \langle u, d - y_T - y_H \rangle \end{aligned}$$

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DUAL maxmin replaces minmax

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Can we have both (x, y) and u???

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Need **primal-dual** scissors

Primal-dual scissors: Augmented Lagrangians

$$L_r(x,y,u) = L(x,y,u) + \frac{r}{2}\|d - y_T - y_H\|^2$$

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$$L^{\#}(x,y,u,\boxed{r}) = L(x,y,u) + r|d - y_T - y_H|_1$$

$\boxed{r}$ is a dual variable

Generalized Augmented Lagrangians (GAL)

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Generalized Augmented Lagrangians (GAL)

The perturbation function p

$$p(u) = \left\{ \begin{array}{ll} \inf & f_T(x, y_T) + f_H(y_H) \\ \text{s.t.} & (x, y_T) \in \mathcal{S}_T \,, y_H \in \mathcal{S}_H \\ & y_T + y_H = d + u \end{array} \right. \quad \Longleftrightarrow \quad p(u) = \left\{ \begin{array}{ll} \inf & \varphi(x) \\ \text{s.t.} & x \in X \\ & h(x) = u \end{array} \right.$$

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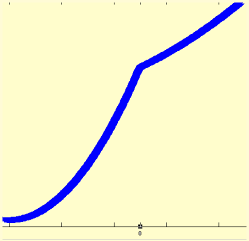
If in the example
$f_{T/H}$ are quadratic,
p is the minimum
of two quadratic functions



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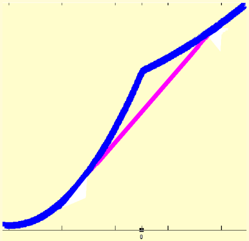

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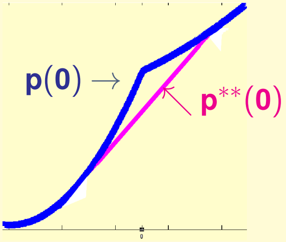

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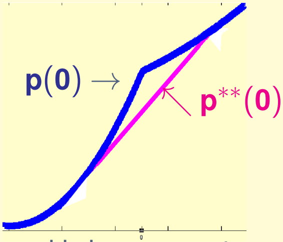

Need a “wedge” to close duality gap and bring magenta curve closer to blue one

The GAL according to R&W

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- p as a marginal function of $\mathcal{D}$

The GAL according to R&W

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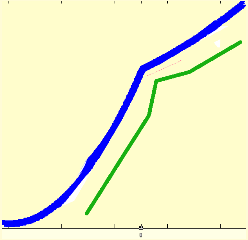

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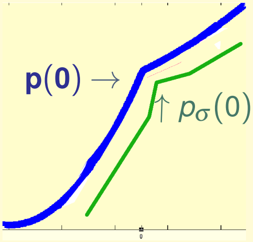

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no duality gap!

Sharp and Proximal GAL according to R&W

$$L_\sigma(x,u,r)$$

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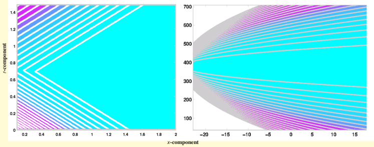

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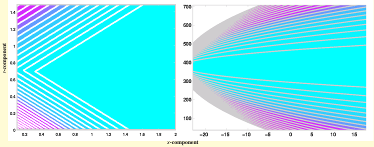

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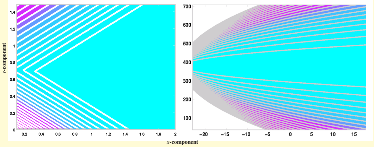

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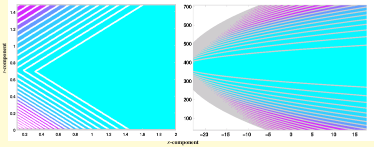

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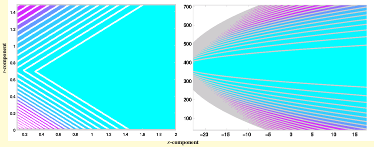

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Our work — eprints for the optimization community — **Revisiting Augmented Lagrangian Duals**

is a proposal to address these issues

GAL according to

Revisiting Augmented Lagrangian Duals

$$\text{GAL of}\ \begin{cases} \min & \varphi(x) \\ \text{s.t.} & x \in X \\ & h(x) = 0 \end{cases}$$

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From now on we consider $L(x, u, r)$ for problem on the right

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$\boxed{\text{Any CQ for original problem yields multipliers } (u,r) \text{ for the } \sigma\text{-augmented problem}}$

Everett's theorem

σ **continuous, non-negative with unique minimizer at 0**

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Fix $(\tilde{u}, \tilde{r})$ and consider evaluating $\theta(\tilde{u}, \tilde{r}) = \min_x L(x, \tilde{u}, \tilde{r})$,

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  - Augmentation (approximately) closes the duality gap, provided $h(\tilde{x}) = 0$

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- Summing up:
  - Augmentation (approximately) closes the duality gap, provided $h(\tilde{x}) = 0$
  - $h(x) = 0 \iff \sigma(h(x)) = 0$
  - An inexact bundle method drives $\sigma(h(x^k))$ to 0

GAL and the σ-simplex

recall that $h(x) = 0 \iff \sigma(h(x)) = 0$

$$\theta(\tilde{u},\tilde{r}) = \min_{x\in X} L(x,\tilde{u},\tilde{r}) = \min_{x\in X} \varphi(x) + \langle \tilde{u}, h(x)\rangle + \tilde{r}\sigma(h(x))$$
$$\Longrightarrow \Big(h(x^{min}), \sigma(h(x^{min}))\Big) \in \partial_\varepsilon \theta(\tilde{u},\tilde{r})$$

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Suppose X compact

Explicit calculus rule in

ε-Optimal solutions in nondifferentiable convex programming and some related questions

J. -J. Strodiot, V. Hien Nguyen & Norbert Heukemes

Mathematical Programming **25**, 307–328(1983)

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$\Longrightarrow \sigma$-simplex=conv$\left\{ \sigma(h(x^i)),\ x^i \text{ approximate minimizers} \right\}$

Theorem: $0 \in \sigma$-simplex is equivalent to

- $0 \in \partial_\varepsilon \theta(\tilde{u}, \tilde{r})$
- One $x^{i_{best}}$ in the σ-simplex is primal feasible: $h(x^{i_{best}}) = 0$

$\Longrightarrow x^{i_{best}}$ approximate solution for original problem

PDBM: a primal-dual bundle method for GAL

Init Choose (u^1, r^1) and compute $x^1 \approx \min_{x \in X} L(x, u^1, r^1)$

Dual Solve bundle QP with a model for θ to obtain (u^+, r^+)

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Theorem:

1. Noise attenuation loop is finite
2. There is a primal feasible limit point $\bar{x}^{i_{best}}$ that is approximately optimal
3. If the dual sequence has accumulation points, they solve approximately the dual problem
4. For existence of dual accumulation points, see “Convex proximal bundle methods in depth” MP2014

Solving difficult problems with PDBM for GAL

DC problems with explicit nonconvexity (exact solution of subproblems)

$$\begin{cases} \min\limits_{x \in X = \mathbb{R}^n} & \varphi(x) := \frac{1}{2}\langle x, Qx\rangle + \langle x, q\rangle - \max\limits_{i \in \{1,\ldots,N\}} \{\langle x, \alpha_i\rangle + \beta_i\} \\ \text{s.t.} & h(x) := Ax - b = 0 \end{cases}$$

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Each θ_i is a QP having the same quadratic term for all i

$$\theta_i(u,r) = \min_x \frac{1}{2}\langle x, Qx\rangle + \langle x, q\rangle + \langle x, \alpha_i\rangle + \beta_i + \langle Ax - b, u\rangle + \frac{1}{2} r\|Ax - b\|_2^2$$

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	PDBM		MSM	(Gas02)	ENUM	
	avg	stdev	avg	stdev	avg	stdev
$\Delta\varphi$	-1E-03	3E-03	-7E-02	3E-01	0	0
$h(\bar{x})$	4E-08	3E-08	1E-03	6E-03		
#primal	25	42	105	115		
CPU (s)	5	8	21	36	209	515

Solving difficult problems with PDBM for GAL

Unit-commitment problems (inexact solution of subproblems, using ADMM)

$$
\left\{
\begin{array}{ll}
\min & \displaystyle\sum_{i \in I} \Big(\langle \mathcal{F}, x_i \rangle + C_i(y_i) \Big) \\
\text{s.t.} & (x_i, y_i) \in \mathcal{S}_i \\
 & \displaystyle\sum_i y_i = D
\end{array}
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\iff
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 & \sum_i z_i = D\\
 & h(x,y) = z - y = 0
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$$

augmented with $\sigma(\cdot) = |\cdot|_1$ gives

$$
L(x, y, z, u, r) = L_{0\text{-}1}(x, y, u) + L_{cont}(z, u) + r|y - z|_1
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augmented with $\sigma(\cdot) = |\cdot|_1$ gives

$$\begin{aligned} L(x,y,z,u,r) &= L_{0\text{-}1}(x,y,u) + L_{cont}(z,u) + r|y-z|_1 \\ &\approx \underbrace{L_{0\text{-}1}(x,y,u) + \frac{r}{2}|y - z^{fixed}|_1} \quad + \quad \underbrace{L_{cont}(z,u) + \frac{r}{2}|y^{fixed} - z|_1} \end{aligned}$$

Solving difficult problems with PDBM for GAL

Unit-commitment problems (inexact solution of subproblems, using ADMM)

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Solving difficult problems with PDBM for GAL

Unit-commitment problems (inexact solution of subproblems, using ADMM)

Very good performance, provided r_0 **is well chosen** (not too large)

Results for 56 synthetic instances, horizon from 1 to 7 days, hourly discretization

	PDBM		MSM	(Gas02)
	avg	stdev	avg	stdev
$Gap(\%)$	4.5	6.5	14.6	12.1
$h(\bar{x})$	4.5E-03	7.2E-04	5.3E-03	1.1E-03
#primal	105	52	208	131
CPU (s)	96	121	148	140

Some works on GAL

- E. Golshtein and N. Tretyakov, Modified Lagrangians and Monotone Maps in Optimization, Wiley **1996**
- R. Rockafellar and R. Wets. Variational Analysis. Springer, **1998**
- A. M. Rubinov, B. M. Glover and X. Q. Yang, "Decreasing Functions with Applications to Penalization", **SiOPT 1999**
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Questions?