Static linear feedback for control as optimization problem

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Goals

- Links between control and optimization

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- Nonconvex optimization

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- Nonconvex optimization
- Nonstandard step-size

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- Links between control and optimization
- Nonconvex optimization
- Nonstandard step-size
- Numerous extensions and applications

Part I

Linear Quadratic Regulator

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

$$\int_0^\infty \left[x(t)^T Qx(t) + u(t)^T Ru(t)\right] dt \to \min$$

$$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, Q \succ 0, R \succ 0$$

A, B controllable.

Explicit solution

Kalman 1960

$$u(t) = -Kx(t), \quad K \in \mathbb{R}^{m \times r}$$

Static linear feedback (vs open-loop control $u(t)$)

$$K = R^{-1}B^T X$$

$X \succ 0$ is the solution of ARE

$$A^T X + XA - XBR^{-1}B^T X + Q = 0$$

Feedback optimization

$$\dot{x}(t) = A_K x(t), \quad A_K = A - BK$$

$$f(K) = \int_0^\infty \left[x(t)^T (Q + K^T R K) x(t)\right] dt \to \min$$

Assume $K_0 \in S$ is known,

$$S = \{K : A_K \text{ is Hurwitz stable}\}$$

Main tools

- Lyapunov lemma: $A^T P + PA = -G, \quad G \succ 0$ has the solution $P \succ 0$ iff A is Hurwitz

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- Bellman lemma: If A is Hurwitz, $\dot{x} = Ax$, then $\int_0^\infty x^\top(t) W x(t) dt = x_0^\top X x_0$, where $X \succ 0$ is the solution of Lyapunov equation $A^\top X + XA = -W$

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- Averaging over $x(0)$: $\mathbb{E} x(0) x(0)^\top = \Sigma$

LQR as matrix optimization problem

$$f(K) = \mathrm{Tr}\,(X\Sigma) \to \min_{K \in S}$$

$$(A - BK)^\top X + X(A - BK) + K^\top RK + Q = 0,$$

$$S_0 = \{K \in S : f(K) \leq f(K_0)\}$$

Goal: apply gradient-like methods for the minimization of $f(K)$.

References

Kalman 1960

Levine, Athans 1970

. . .

J. Bu, A. Mesbahi, M. Fazel, M. Mesbahi, ArXiv:1907.08921, 2019,

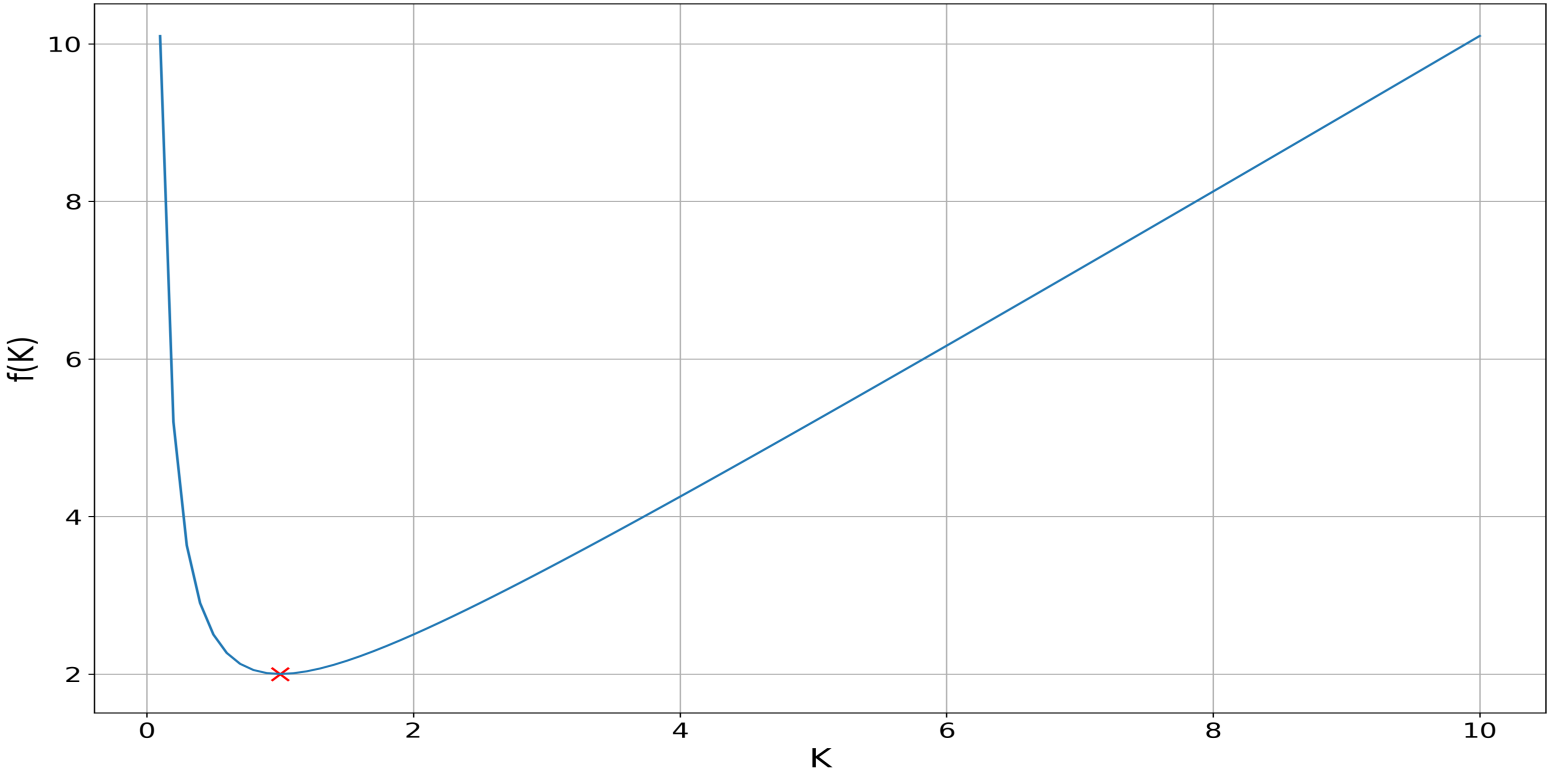
M.Fazel, R.Ge, S.Kakade, M.Mesbahi, ICML, 2018 - discrete-time.

H.Mohammadi, A.Zare, M.Soltanolkotabi, M.Jovanovic, ArXiv:1912.11899, 2019, J. Bu, A. Mesbahi, M. Mesbahi, ArXiv:2006.09178, 2020

I.Fathullin, B.Polyak, ArXiv:2004.09875, 2020, submitted to SIAM J. Contr. Opt.

Example 1

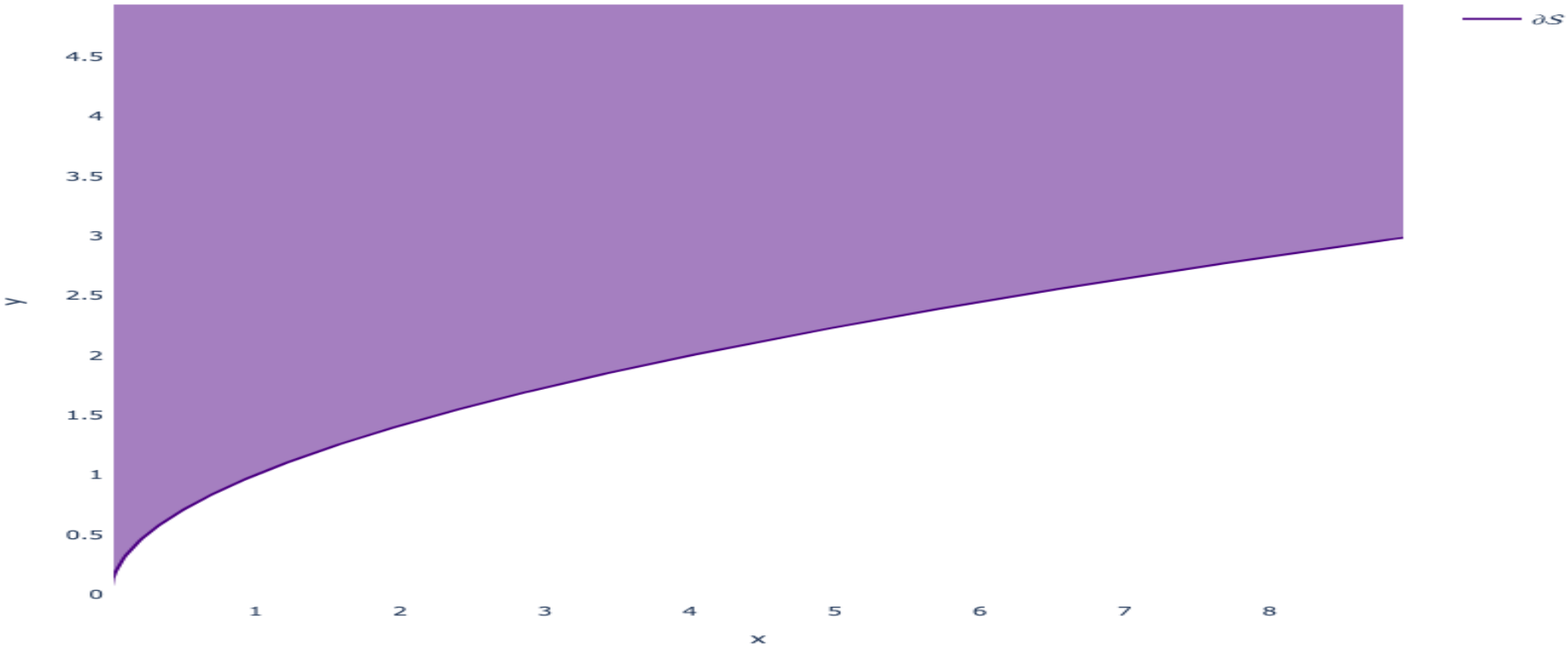
$$n = 1, A = 0, 2B = Q = R = 1, f(K) = K + 1/K$$



$S = R^1_+$ is unbounded, $f(K)$ grows near the boundary.

Example 2

$n = 3, m = 1, S : k_1 > 0, k_2 k_3 > k_1$

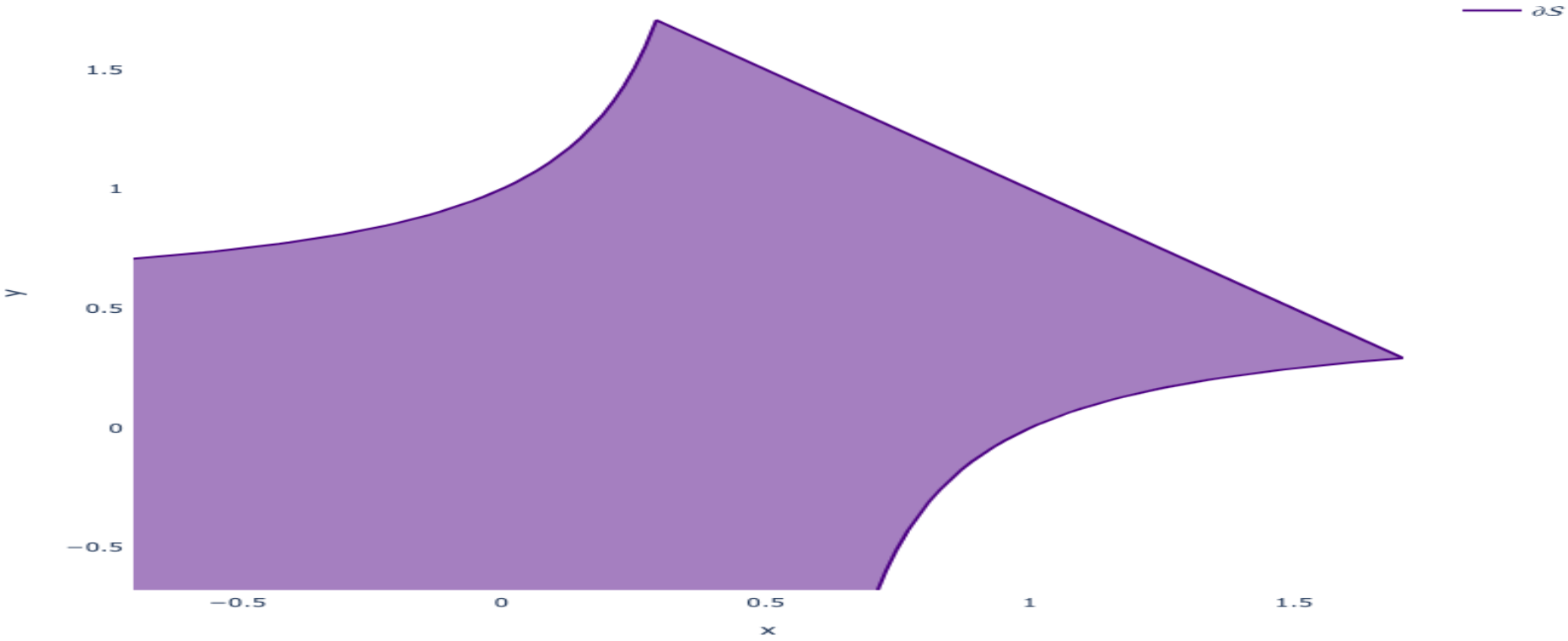


S is non-convex unbounded.

Example 3

$m = n = 2, A = B = Q = R = I$

$S : k_{11} + k_{22} < 1 + k_{11}k_{22} + k_{12}k_{21}, k_{11} + k_{22} < 2$



S is non-convex

Properties of $f(K)$ and S: Connectedness of S, S_0

Lemma: The sets S and S_0 are connected.

Proof: change of variables $P = X^{-1}$ reduces the problem to a convex one, while stabilizing K have the form $K = R^{-1}B^T P^{-1}$, this continuous image of the convex set $P \succ 0$ is connected.

Change of variables allows to reduce LQR to SDP H.Mohammadi, A.Zare, M.Soltanolkotabi, M.Jovanovic, ArXiv:1912.11899, 2019. We avoid this trick.

Properties of $f(K)$ and S: $f(K)$ is coercive

Lemma: $f(K_j) \to +\infty$ for $\|K_j\| \to +\infty$ or $K_j \to K \in \partial S$

It follows from the estimates of solutions of the Lyapunov equation.

Properties of $f(K)$ and S: S_0 is bounded

Lemma: S_0 is bounded

It follows from the previous lemma.

Properties of $f(K)$ and S: $f(K)$ is differentiable

Kalman 1960, Levine-Athans 1970

$$\nabla f(K) = 2\left(RK - B^T X\right) Y,$$

$$A_K^\top X + XA_K + K^\top RK + Q = 0,$$

$$A_K Y + YA_K^\top + \Sigma = 0, \quad A_K = (A - BK).$$

The minimizer K_* exists, $\nabla f(K_*) = 0$. This implies ARE for K^*.

Properties of $f(K)$ and S: $f(K)$ is twice differentiable

$$\frac{1}{2}\nabla^2 f(K)[E,E] = \langle REY, E\rangle - 2\left\langle B^\top X' Y, E\right\rangle$$

$$A_K^\top X' + X' A_K + M^\top E + \left(M^\top E\right)^\top = 0$$

$$M = RK - B^\top X$$

Properties of $f(K)$ and S: $f(K)$ is L-smooth on S_0

Lemma: The objective $f(K)$ is L-smooth on S_0 with $f(K_0)$-dependent constant L.
L can be large for K close to the stability boundary. In Example 1 $f(K) = K + 1/K$ and $f''(K) = 2/K^3$. Thus $f(K)$ is not L-smooth on S.

Properties of $f(K)$ and S: Gradient domination

Lemma: Condition LPL holds on S_0:

$$f(K) - f_* \leq \mu ||\nabla f(K)||^2, \quad \mu > 0$$

Lemma: $f(K)$ is strongly convex in the neighborhood of K_*.

Gradient flow

$$\dot{K}(t) = -\nabla f(K), \quad K(0) = K_0 \in S$$

Theorem: Solution of ODE $K(t)$ exists and $\lim_{t\to\infty} K(t) = K_*$

Need to solve two Lyapunov equations for every t.

Gradient descent

$$K_{j+1} = K_j - \gamma_j \nabla f(K_j)$$

Step-size choice

- $K_{j+1} \in S_0$
- Monotonicity
- Armijo-like condition

  Such algorithms can be implemented due to properties of $f(K)$.

Theorem: $\lim_{j\to\infty} K_j = K_*$ with linear rate.

Alternative step-size for GD

$$\min f(x), \quad x \in R^n$$

$$x_{j+1} = x_j - \gamma_j \nabla f(x_j), \gamma_j = \frac{||\nabla f(x_j)||^2}{(\nabla^2 f(x_j) \nabla f(x_j), \nabla f(x_j))}$$

This is Newton method for 1D minimization. For $f(x)$ quadratic, the method coincides with steepest descent.

Theorem: For $f(x)$ L-smooth and μ-strongly convex, the method locally converges, whereas its damped version converges globally.

Example $f(x) = \frac{1}{x} + x, \quad x_0 > 0$ small, $x_1 \approx \frac{3}{2} x_0$.

A version of conjugate gradient

$$\min f(x), \quad x \in R^n$$

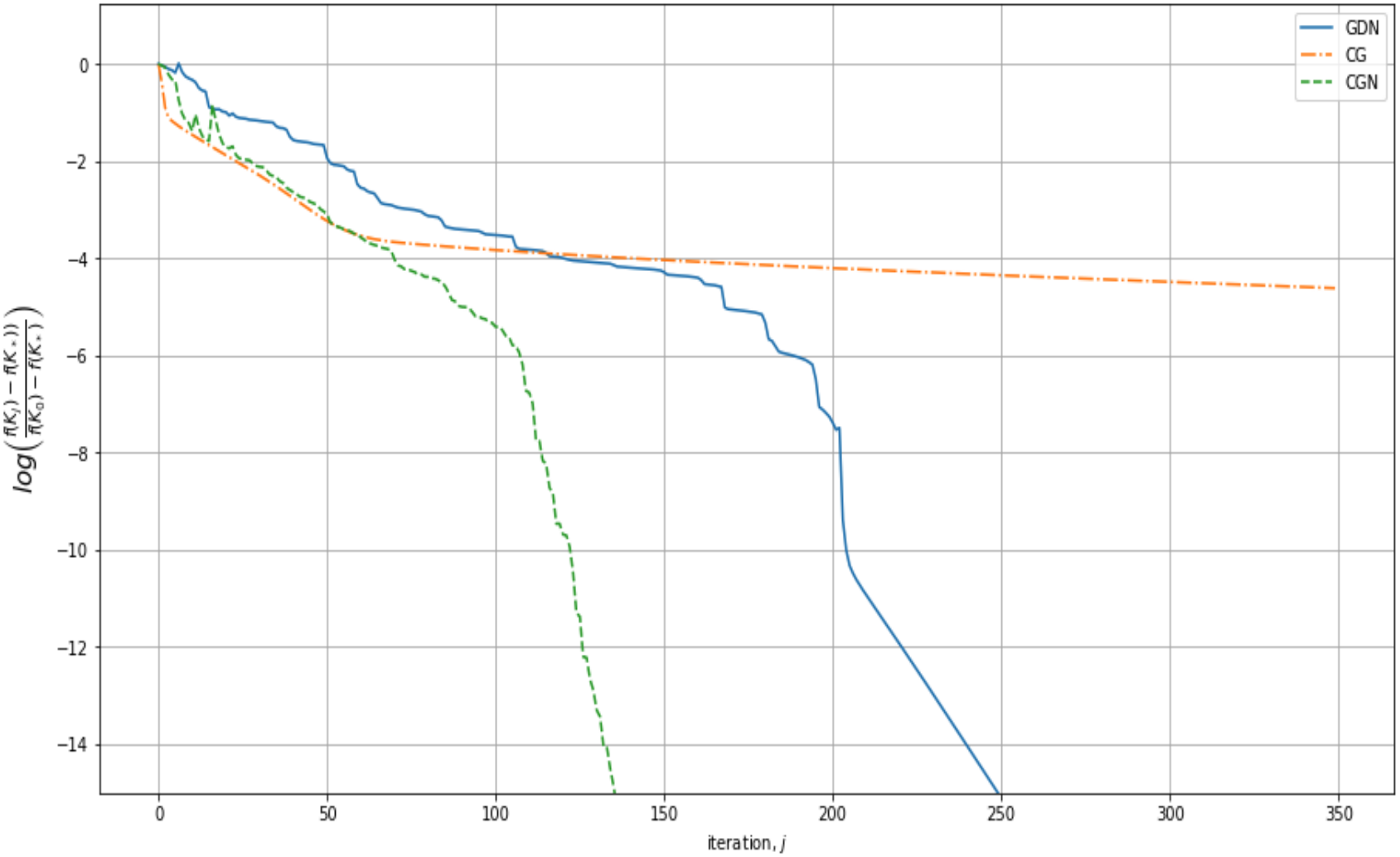
$$x_{j+1} = x_j + \alpha_j p_j, \; \alpha_j = \frac{(\nabla f(x_j), p_j)}{(\nabla^2 f(x_j) p_j, p_j)}$$

$$p_j = -\nabla f(x_j) + \beta_j p_{j-1}$$

$$\beta_j = \frac{||\nabla f(x_j)||^2}{||\nabla f(x_{j-1}||^2}, \beta_0 = 0.$$

Simulation

Random data, $n = 50, m = 10, K_0 = 0$



Reduced gradient (RG)

Wolfe 1968

$$\min_{x,y} f(x,y), \quad g(x,y) = 0$$

Assume equation $g(x,y) = 0$ has the solution $x(y)$ for all $y \in S$. For $F(y) := f(x(y), y)$ arrive at unconstrained optimization

$$\min F(y), \quad y \in S$$

It is not hard to find the gradient of $F(y)$ and apply gradient descent; this is RG method. We are in this framework with $x = X, y = K$. Thus global convergence of RG can be validated.

Extensions

- Discrete-time case.

  J. Bu, A. Mesbahi, M. Fazel, M. Mesbahi, ArXiv:1907.08921, 2019;

  M.Fazel, R.Ge, S.Kakade, M.Mesbahi, ICML, 2018.
- Finite horizon case, Large-scale problems, Implementation issues.
- Application for MPC.
- Relatively-smooth functions

  H.Lu, R.Freund, Y.Nesterov, ArXiv:1610.05708, 2017

  H.Bauschke, J.Bolte, M.Teboulle, MOR, 2017.

Part II

Optimization via Low Order Controllers

$$\dot{x}(t) = A(k)x(t), \quad A(k) = A_0 + \sum_{i=1}^{m} k_i A_i,$$

$$\min \int_0^\infty x(t)^T Q x(t) dt + \sum_{i=1}^{m} \gamma_i k_i^2$$

$$x(0) = x_0, A_i \in \mathbb{R}^{n \times n}, Q \succ 0, \gamma_i > 0.$$

Quadratic term is the penalty for excessive control.

Particular cases

- Static output feedback

$$y = Cx, \quad u = Ky, \quad A(K) = A + BKC$$

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$$y = Cx, \quad u = Ky, \quad A(K) = A + BKC$$

- Decentralized control $u = Kx, \quad K \in L$
- PID controllers

Optimization setup

$$f(k) := \mathrm{Tr}\,(P) + \sum_{i=1}^{m} \gamma_i k_i^2 \to \min_{k \in \mathcal{S}},$$

$$A(k) = A_0 + \sum_{i=1}^{m} k_i A_i.$$

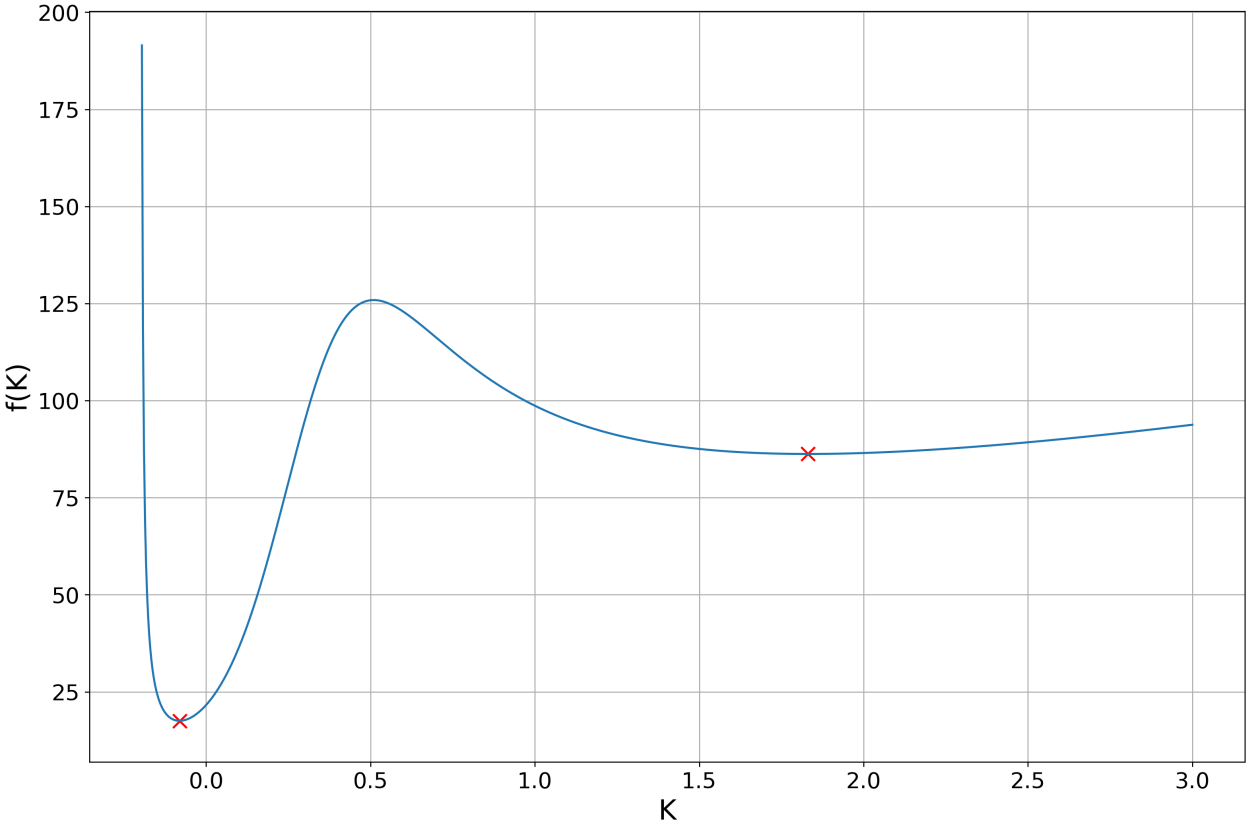
$$A(k)^\top P + PA(k) = -Q,$$

$$S = \{k : A(k) \text{ is Hurwitz}\},$$

$$k^{(0)} \in S \text{ is known.}$$

Example 1

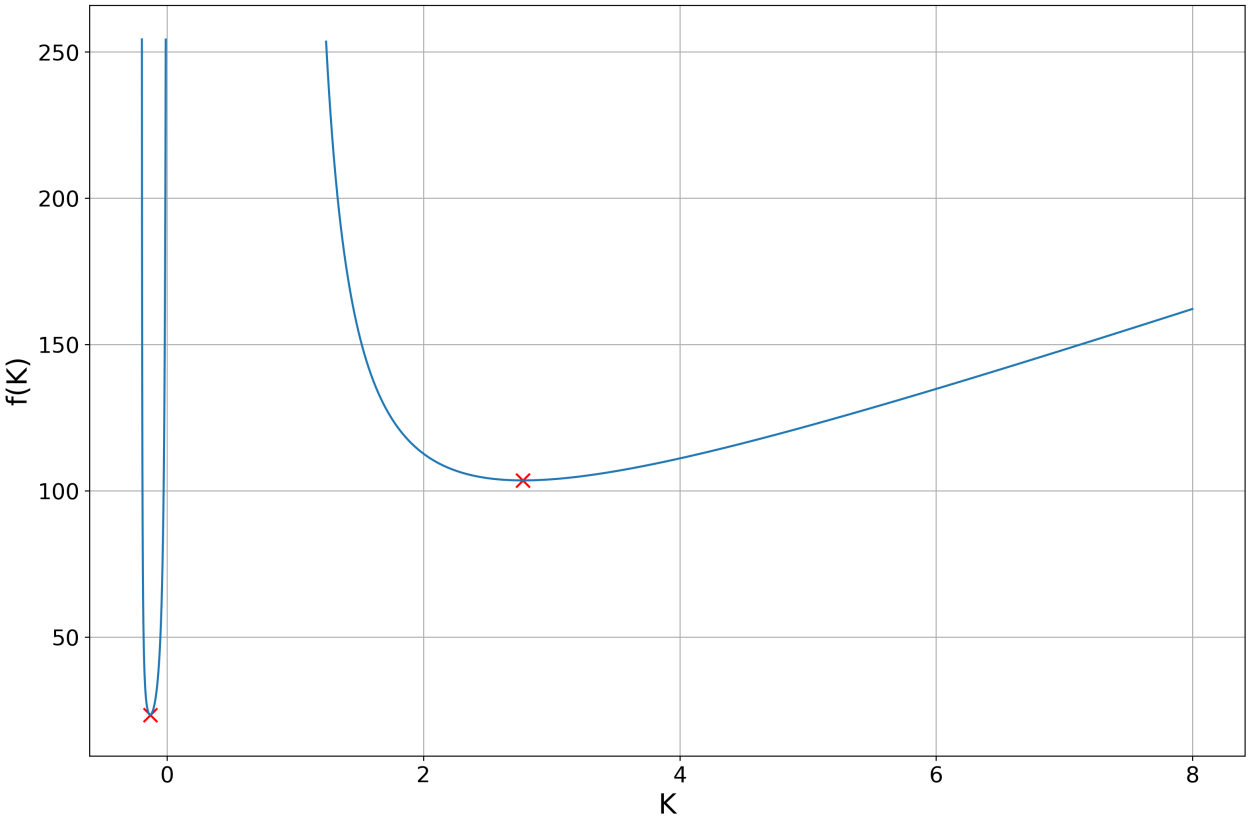
$n = 3, m = 1$



Several local minima

Example 2

$n = 3, m = 1$



Several local minima. Non-connected S

Decentralized control

H. Feng, J. Lavaei, ACC 2019

$$A(k) = \begin{pmatrix} -1 & 2+k_1 & 0 & 0 \\ -1-k_1 & 0 & 2+k_2 & 0 \\ 0 & -1-k_2 & 0 & 2+k_3 \\ 0 & 0 & -1-k_3 & 0 \end{pmatrix}.$$

Stability region S for system of order n may have 2^{n-1} connectivity components.

Properties of $f(k)$ and S

- Growth near boundary: $f\left(k^{(j)}\right) \to +\infty$ when $\left\|k^{(j)}\right\| \to +\infty$ or $k^{(j)} \to k \in \partial S$.
- Bounded $S_0 = \{k \in S : f(k) \leq f(k^{(0)})\}$.
- Number of connectivity components of S can be large.

Derivatives of $f(k)$

$$\nabla_i f(k) = \mathrm{Tr}(P^i) + 2\gamma_i k_i,$$
$$\nabla^2_{ii} f(k) = \mathrm{Tr}(P^{ii}) + 2\gamma_i,$$

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$$\nabla^2_{ii} f(k) = \mathrm{Tr}(P^{ii}) + 2\gamma_i,$$

where P, P^i and P^{ii} satisfy

$$A(k)^\top P + PA(k) = -Q,$$

$$A(k)^\top P^i + P^i A(k) = -\left((A_i)^\top P + PA_i\right),$$

$$A(k)^\top P^{ii} + P^{ii} A(k) = -2\left((A_i)^\top P^i + P^i A_i\right).$$

Properties of $f(k)$

$f(k)$ is L—smooth on S_0 with $f(k^{(0)})$-depending constant L.

Unfortunately gradient domination is lacking.

Algorithms

- Gradient descent
- Coordinate descent

Theorem: For smart versions of the algorithms $f(k_{j+1}) \leq f(k_j)$, and $\nabla f(k_j) \to 0, \quad j \to \infty.$

Example - Low order controller

A.Krasovsky, 1967

$$n = 3, m = 2:$$

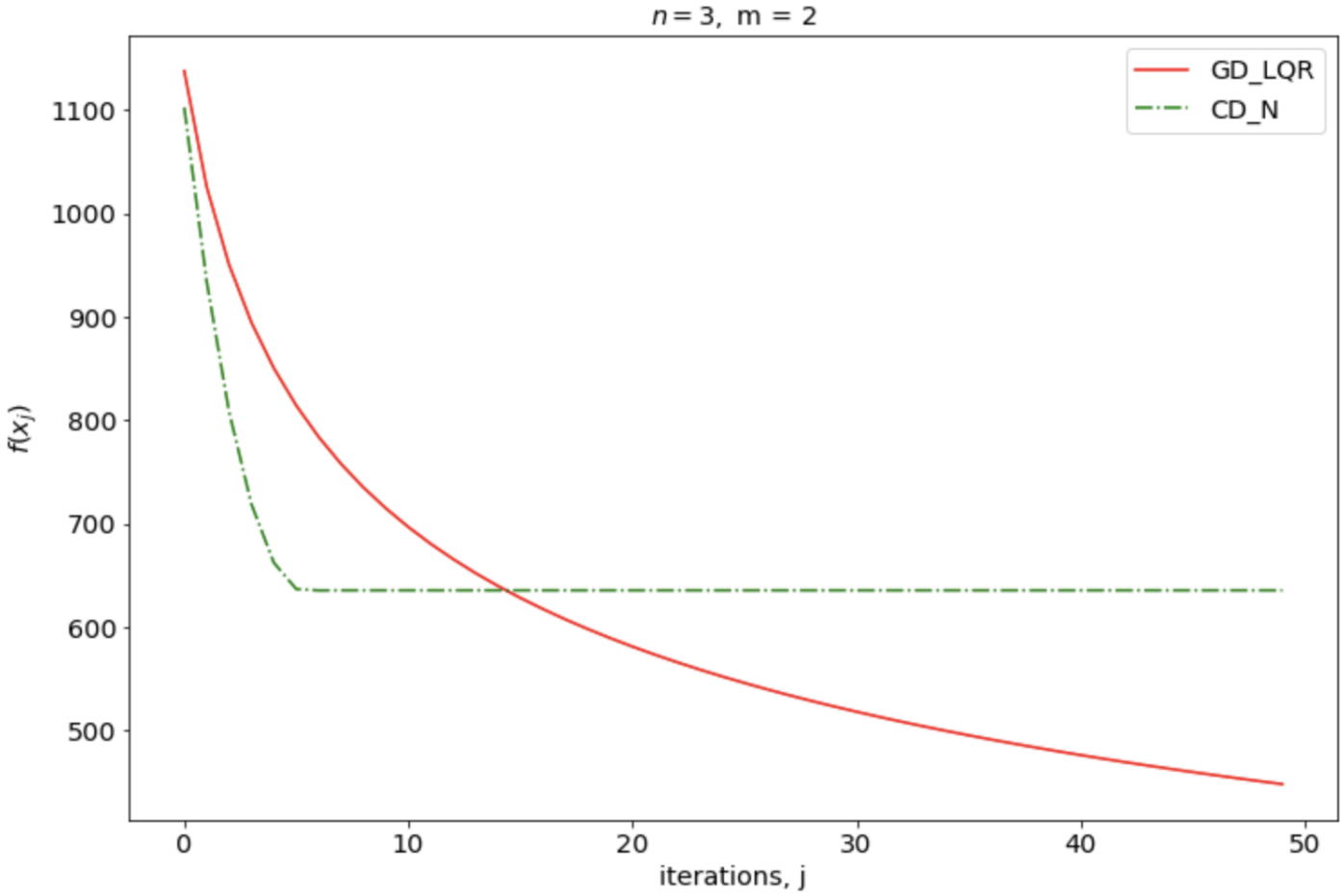
$$A(k) = \begin{pmatrix} -1 & 0 & -k_1 \\ k_2 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\gamma_1 = \gamma_2 = 1,$$

$$k^{(0)} = (1, 1).$$

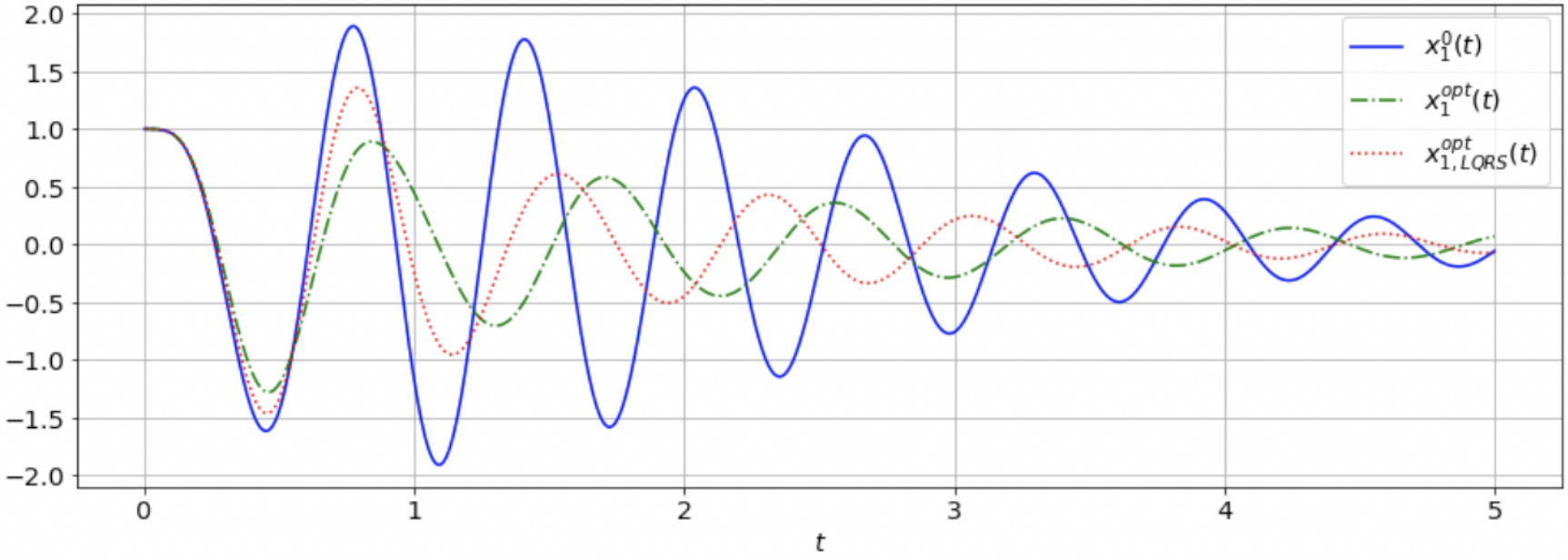
Example - Simulation

Low-order controller vs LQR



Performance for low-order controller is worse, but convergence is faster.

Example - Transient response



Conclusions on low-order controllers

- Fears of global optimization are exaggerated
- Moreover, our goal is to improve the initial controller, not to find the best one.

Part III

Extensions

- Alternative objective functions

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- Nonlinear matrix inequalities

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- Linearization (=trust-region) methods

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- Alternative objective functions
- Nonlinear matrix inequalities
- Linearization (=trust-region) methods
- Systems with disturbances

Peak effect

Discrete-time system, A is Schur stable matrix,

$$x_{k+1} = Ax_k, \quad \eta(A) = \max_{|x_0|=1} \max_k |x_k| = \max_k ||A^k||$$

Upper bound: $\eta(A) \leq \eta_{\mathrm{upp}}(A) = ||Q||^{1/2}$, Q being the solution of the SDP

$$\min ||Q||, \quad A^T QA - Q \succ 0, \quad Q \succ I$$

Lyapunov function $V(x) = (Qx, x)$, invariant ellipsoid $V(x) \leq V(x_0)$.

Reduction of peak

State feedback

$$x_{k+1} = Ax_k + Bu_k, \quad u_k = Kx_k$$

Minimize $\eta_{\mathrm{upp}}(A + BK)$. Via change of variables $P = Q^{-1}, Y = KP$ this can be converted to SDP in P, Y.

Static output feedback

$y = Cx, \quad u = Ky$. Then the problem cannot be reduced to convex optimization and we arrive at nonlinear matrix inequalities in variables P, K, γ

$$\min \gamma, \quad (A + BKC)P(A + BKC)^\top - P \prec 0,$$

$$I \preccurlyeq P \preccurlyeq \gamma I$$

with an upper bound for peak $\eta_{\mathrm{upp}}(A + BKC) = \gamma^{1/2}$.

Trust region method

Linearize BMI at a point P, K with $A(K) = A + BKC$:

$$A(K)(P + \delta P)A(K)^\top + A(K)P(B\,\delta K\,C)^\top +$$

$$+(B\,\delta K\,C)PA(K)^\top - P - \delta P \prec 0$$

and solve SDP in $\delta K, \delta P, \gamma$ with this LMI and $\|\delta K\| \leq \varepsilon$, $I \preccurlyeq P + \delta P \preccurlyeq \gamma I$. Adjust ε to guarantee solvability of LMIs and monotonicity of γ.

Example

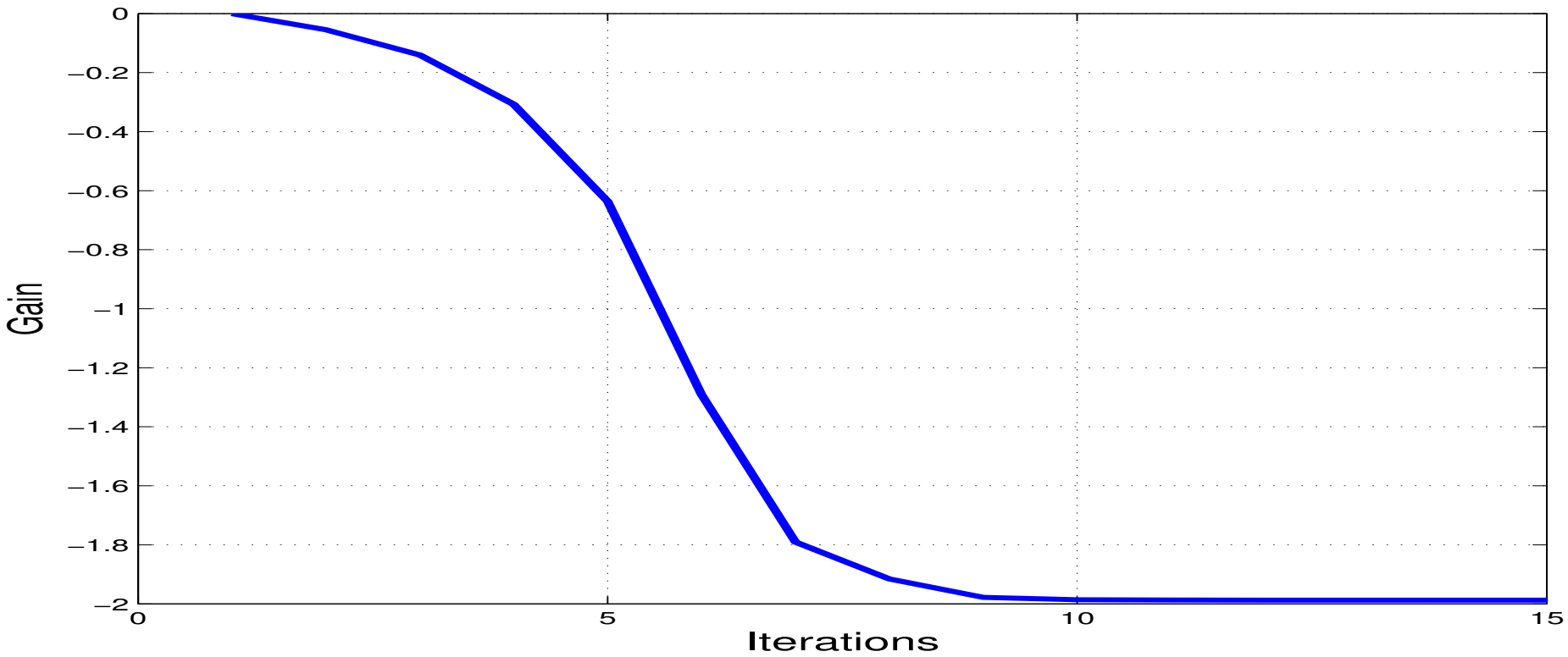
Dowler 2013, Shcherbakov 2017

$$A = \begin{pmatrix} \frac{20}{21} & 2 & 0 & 0 \\ 0 & \frac{40}{41} & 2 & 0 \\ 0 & 0 & \frac{60}{61} & 2 \\ 0 & 0 & 0 & \frac{80}{81} \end{pmatrix}$$

A is stable: $\rho(A) \approx 0.988$, $\|A\| = 2.807$,
$\eta(A) \approx 1.5 \times 10^5$ at $k^* = 141$,
$\eta_{\mathrm{upp}}(A) \approx 1.7 \times 10^5$.
$B = (0 \;\; 0 \;\; 0 \;\; 1)^\top$, $C = (0 \;\; 0 \;\; 0 \;\; 1)$, scalar K (we vary a_{44}).

Example - simulation

$K = -1.9874$, $\eta(A(K)) \approx 2000$ at $k^* = 75$, $\eta_{\mathrm{upp}}(A(K)) \approx 2400$



Validation

Rigorous algorithm and its validation remain open problems. However the results of simulation are promising.

Problems with disturbances

$$\dot{x} = Ax + Bu + Dw, \quad x(0) = x_0$$

$w(t) \in R^m$ is external disturbance. If it is Gaussian, we are in the framework of LQG. In contrast, we assume it non-random and bounded:

$$|w(t)| \leq 1$$

Then integral quadratic objective has no sense, and we deal with invariant ellipsoid E_x. We take state feedback $u = Kx$ and linear output $z(t)$, the goal is to minimize the bounding ellipsoid E_z for the output as function of K.

Invariant ellipsoid

$$\dot{x} = (A + BK)x + Dw, \quad x(0) = x_0$$

$$z = Cx + B_2 u = (C + B_2 K)x$$

Important: z includes u, e.g. $z = [Cx, u]^T$.

Invariant ellipsoid for x: $E_x = \{x : x^T P^{-1} x \leq 1\}$,

$$x(0) \in E_x \Longrightarrow x(t) \in E_x \quad \forall t \geq 0,$$

$$x(0) \notin E_x \Longrightarrow x(t) \to E_x, t \to \infty$$

Optimization setup

Via special change of variables optimization problem can be reduced to SDP S.Nazin, Polyak, Topunov 2007 . This trick does not work for output feedback and low-order controllers, thus we deal with gain K.

Optimization problem:

$$||(C + B_2K)Q^{-1}(C + B_2K)^T|| \to min$$

$$(A+BK)^TQ+Q(A+BK)+\alpha Q+\frac{1}{\alpha}QDD^TQ \preccurlyeq 0,$$

in the matrix variables $Q = Q^T \succ 0$, K and scalar $\alpha > 0$.

Linearization algorithm

Pshenichny 1970

Idea: linearize objective (L_1) and matrix inequalties (L_2) at the current approximation K, Q, α and solve the convex optimization problem

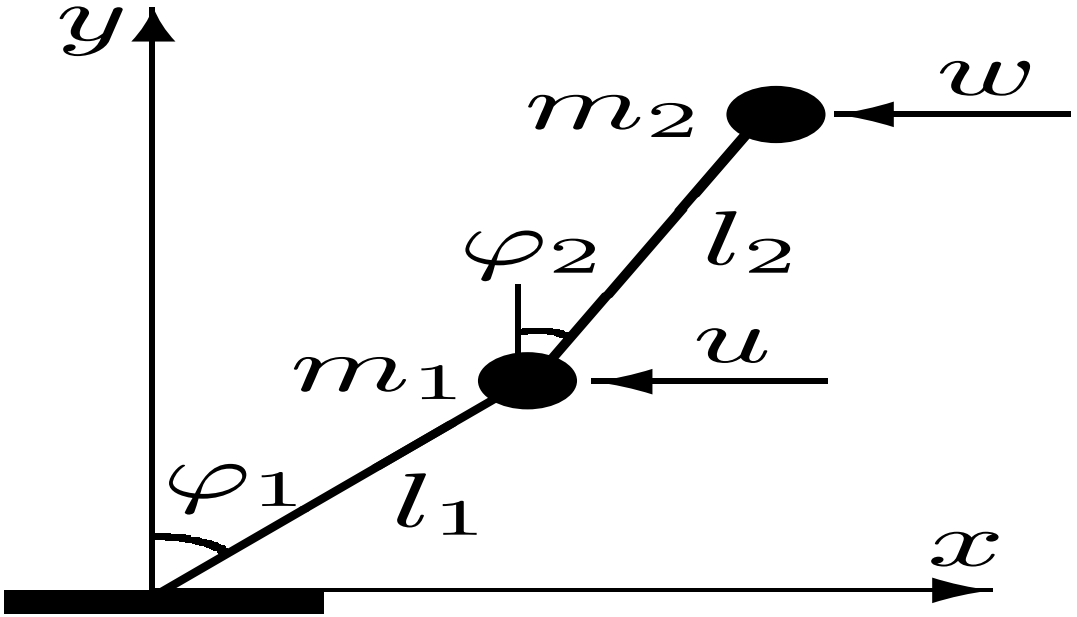
$$L_1 + \frac{1}{\varepsilon_1}\|\delta K\|^2 + \frac{1}{\varepsilon_2}|\delta\alpha|^2 \to \mathbf{min}$$

subject to LMIs

$$L_2 \preccurlyeq 0, \quad Q + \delta Q \succ 0, \quad \alpha + \delta\alpha > 0,$$

with variables $\delta K, \delta Q, \delta\alpha$ and step-sizes $\varepsilon_1, \varepsilon_2$.

Example - double inverted pendulum



$$\dot{\varphi}_1 = \varphi_3,$$
$$\dot{\varphi}_2 = \varphi_4,$$
$$\dot{\varphi}_3 = 2\varphi_1 - \varphi_2 + u,$$
$$\dot{\varphi}_4 = -2\varphi_1 + 2\varphi_2 + w, \quad |w(t)| \leq 1.$$

Example - objective function

State feedback:

$$u = k_1\varphi_1 + k_2\varphi_2 + k_3\varphi_3 + k_4\varphi_4$$

$$z = \begin{pmatrix} \varphi_1 \\ u \end{pmatrix},$$

Minimize trace of the bounding ellipsoid for z.

Example - simulation

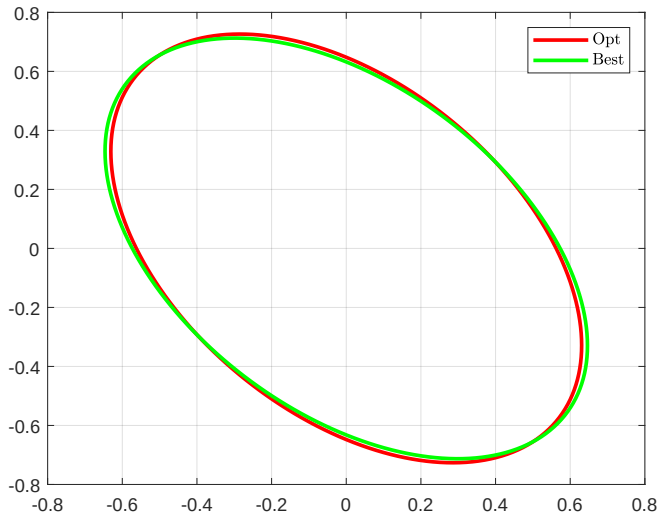
Solution via SDP

$$\widehat{K} = \begin{pmatrix} -24.9103 & 28.9948 & -7.6200 & 19.8484 \end{pmatrix}$$

Linearization method after 20 iterations

$$\widetilde{K} = \begin{pmatrix} -24.6968 & 28.6884 & -7.5712 & 19.6210 \end{pmatrix}.$$

Ellipsoids E_z:



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- Rigorous validation of algorithms is given for some cases, for other cases it remains an open problem
- The results of simulations are promising
- Challenging field for research!