Smoothness in nonsmooth optimization

(Newtonian ideas for partly smooth equations)

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Joint work with: D. Drusvyatskiy, X.Y. Han, A. Ioffe,
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September 2020

ORIE Cornell One World Optimization Seminar

Introduction: three questions

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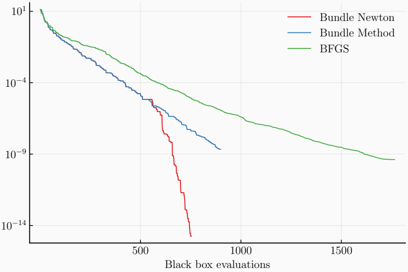
Example: Eigenvalue optimization.



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Example: For random matrices Y, proximal matrices A minimizing

$$\rho(A) + \lambda\|A - Y\|^2$$

often have disc fields of values $\{u^* A u : \|u\| = 1\}$.

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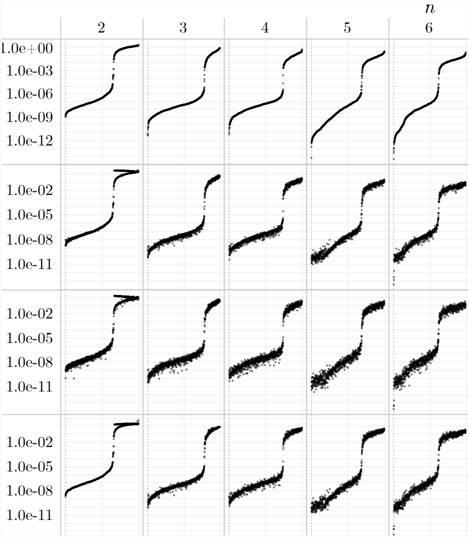

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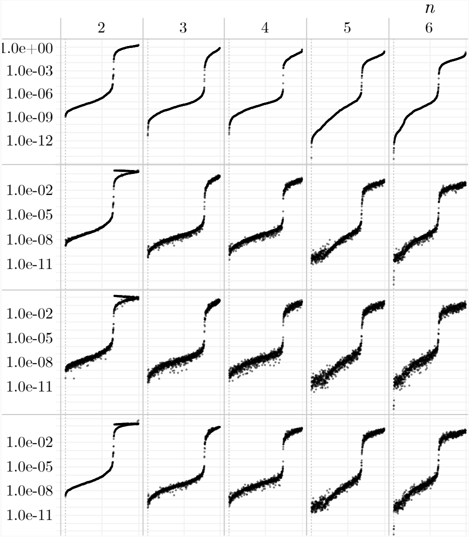

Why. . . ? (Disk matrices comprise a small set, of codimension $2n$).

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History: generalized "active constraints" in nonlinear programming (Burke-Moré '88), "identifiable surfaces" (Wright '93), "$\mathcal{VU}$ decomposition" (Mifflin-Sagastizábal '00)...

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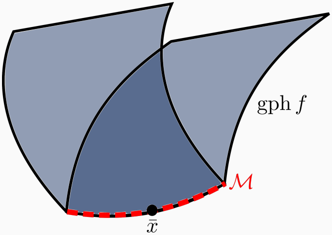

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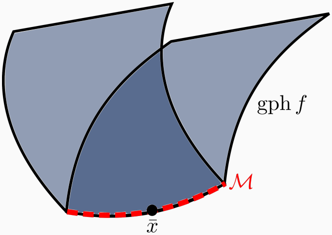

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Example: Nuclear norm regularization for low-rank optimization. Proximal gradient (singular value thresholding) iterates settle on a fixed-rank manifold, then converge linearly to the solution.

(Liang-Fadili-Peyré '18)

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  Find $x \in Q$ so $F(x)^T(z-x) \geq 0$ for all $z \in Q$: equivalently,

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- Composite optimization
  $\min_x h\big(c(x)\big)$ for convex h on $\mathbf{R}^m$ and smooth c into $\mathbf{R}^m$.
  Stationarity:
  $$0 \;\in\; \begin{pmatrix} \nabla c(x)^T y \\ -y \end{pmatrix} + \begin{pmatrix} 0 \\ \partial h(c(x)) \end{pmatrix}$$

Partly smooth generalized equations (L-Liang '18)

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- $\operatorname{gph}\Phi = \{(u, v) : v \in \Phi(u)\}$ is a manifold around $(\bar{u}, \bar{v})$.
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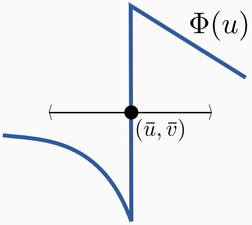

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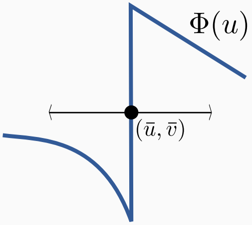

Asymptotic solvers then identify the active manifold

$$\mathcal{M} \;=\; \operatorname{proj}\big(\operatorname{gph}\Phi \text{ around } (\bar{u}, \bar{v})\big),$$

since $v_k \in \Phi(u_k)$ with $(u_k, v_k) \to (\bar{u}, \bar{v})$ implies $u_k \in \mathcal{M}$ eventually.

Basic example: partly smooth sets

For closed convex (or "prox-regular") $S \subset \mathbf{R}^n$, suppose $\bar{x}$ solves

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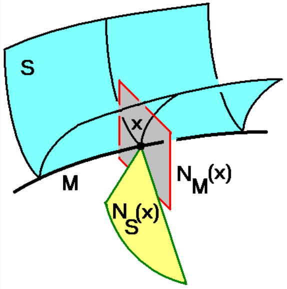
If S contains a ridge manifold $\mathcal{M}$ (the normal cone $N_S(x)$ depends on $x \in \mathcal{M}$ continuously and spans $N_{\mathcal{M}}(x)$), and nondegeneracy holds $\big(\bar{y} \in \mathrm{ri}\big(N_S(\bar{x})\big)\big)$, then the operator N_S is partly smooth at $\bar{x}$ for $\bar{y}$, with active manifold $\mathcal{M}$.



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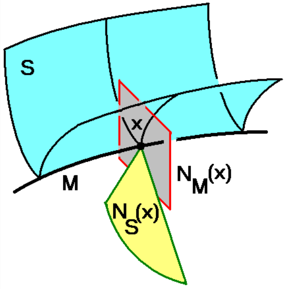

So if S is convex and $\bar{x}$ is unique, projected gradient iterations $x \leftarrow \mathrm{proj}_Q(x - \alpha\bar{y})$ converge to $\bar{x}$ (if α small) and identify $\mathcal{M}$.

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has a unique solution, which furthermore has each $\lambda_i > 0$. Then

$$x \;\mapsto\; \partial f(x) \;=\; \operatorname{conv}\{\nabla f_i(x) : f_i(x) = f(x)\}$$

is partly smooth at $\bar{x}$ for 0 relative to the active manifold $\mathcal{M}$ of points where each f_i has equal value.

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Φ is partly smooth for y at each solution $x_i = G_i(y)$, and

$$\mathrm{gph}\,\Phi \quad \text{intersects} \quad (\mathbf{R}^n\times\{y\}) \quad \text{transversally at } (x_i,y).$$

(Ioffe '07, Bolte...'11, Drusvyatskiy...'16, Lee...'19, L-Tian)

Semi-Newton iterations for $0 \in \Phi(u)$ (L-Wylie '20)

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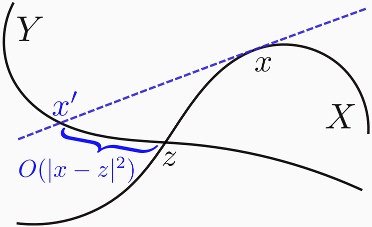

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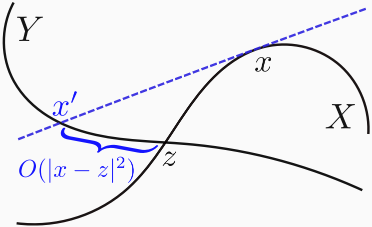

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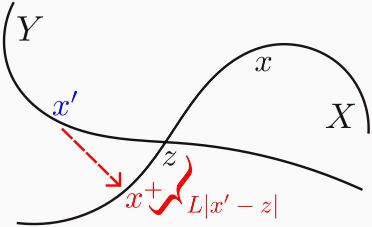

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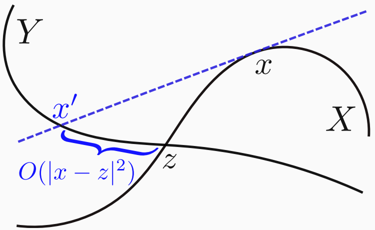

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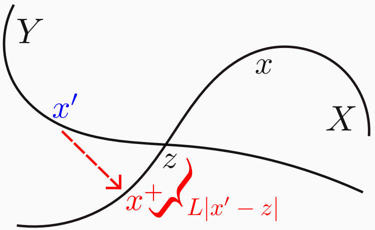

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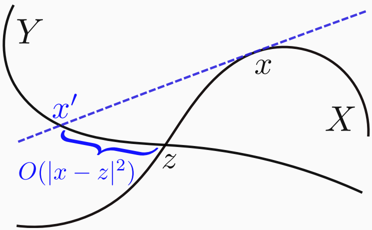

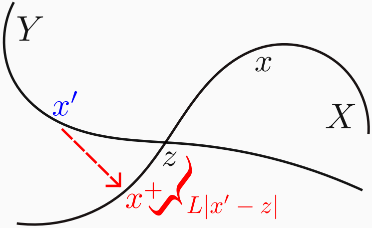

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$u^+ = \mathrm{Proj}_{\mathcal{M}}(u')$; $\quad v^+ = \mathrm{Proj}_{\Phi(u^+)}(0)$

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With just an oracle for **linear** approximations to convex f at input points, bundle methods are appealing (Sagastizábal '18 ICM).

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But can we reduce oracle calls using **quadratic** approximations?

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Intuition: if the bundle size k is large enough,

$$\lim_{S \to \{x\}} \operatorname{conv}\big(\nabla f(S)\big) \;=\; \partial f(x).$$

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For each of the k current reference points $s \in S$, use the oracle to form the linear and quadratic approximations

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- Replace $s \in S$ minimizing $\|\nabla f(s) - \nabla f(x)\|$ with x.

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New point x improves model's most closely matching component.

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The classical Newton method has $k = 1$.

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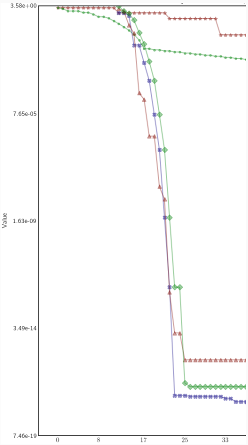
10^{-1}

10^{-5}

10^{-9}

10^{-13}

value



iterations 0 10 20 30

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1

10^{-4}

10^{-8}

10^{-12}

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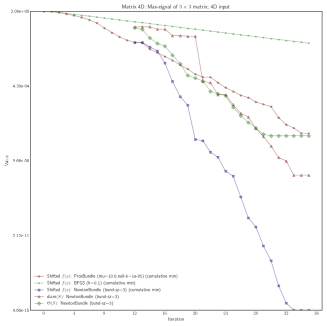

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Each reference point s_i updates within k steps, by strong convexity.

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Black-box methods for finding a minimizer $\bar{x}$ for nonsmooth f, like

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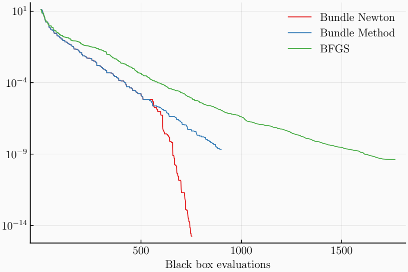

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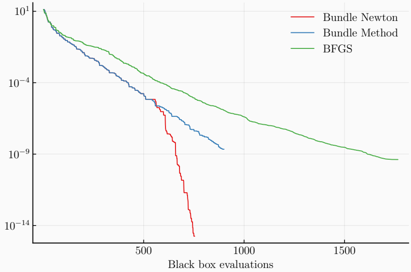
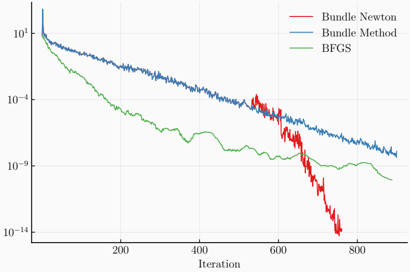


$\lambda_1(A(x)) - \lambda_6(A(x))$



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Take-away...

Partial smoothness is a simple differential-geometric idea that captures the generic interplay between smooth and nonsmooth geometry in concrete variational problems, illuminating the analysis and design of algorithms.

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